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REALIZATION OF GRADED-SIMPLE ALGEBRAS AS LOOP ALGEBRAS

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Dedicated to the memory of Gordon E. Keller, 1939–2003

Abstract. Multiloop algebras determined by n commuting algebra automorphisms of finite order are natural generalizations of the classical loop algebras that are used to realize affine Kac-Moody Lie algebras. In this paper, we obtain necessary and sufficient conditions for a $\mathbb{Z}^n$-graded algebra to be realized as a multiloop algebra based on a finite dimensional simple algebra over an algebraically closed field of characteristic 0. We also obtain necessary and sufficient conditions for two such multiloop algebras to be graded-isomorphic, up to automorphism of the grading group.

We prove these facts as consequences of corresponding results for a generalization of the multiloop construction. This more general setting allows us to work naturally and conveniently with arbitrary grading groups and arbitrary base fields.

1. Introduction

This paper studies the realization, or construction, of graded-simple algebras as loop algebras. Our results are quite general and apply to algebras of any kind including both Lie algebras and associative algebras. However, it was a very specific problem in the theory of infinite dimensional Lie algebras that motivated our work.

In V. Kac's early work on infinite dimensional Lie algebras, he showed that any affine Kac-Moody Lie algebra (or more precisely the derived algebra modulo its centre of any affine Kac-Moody Lie algebra) can be realized as the loop algebra of a finite order automorphism of a finite dimensional simple Lie algebra. This fact is of great importance in the theory of affine algebras. Now extended affine Lie algebras (EALA's) are higher nullity generalizations of affine Kac-Moody Lie algebras [1], and so it is natural to ask if any EALA (or more precisely the centreless core of any EALA) can be realized as the multiloop algebra of a sequence of commuting finite order automorphisms of a finite dimensional simple Lie algebra. Our research on multiloop realizations began with this question.

The centreless core $\mathcal{L}$ of an EALA, now also called a centreless Lie torus ([20, 26]), is in particular graded by a finitely generated abelian group Λ of finite rank n, where n is the nullity of the EALA. Moreover, as a Λ-graded algebra, $\mathcal{L}$ is graded-central-simple. In fact, as our work progressed, it became clear that the methods we were

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employing applied not only to centreless Lie tori but also to graded-central-simple algebras of any kind.

Indeed suppose that Λ is a free abelian group of finite rank and k is an algebraically closed field of characteristic 0. We are able to show in Corollary 8.3.5 that a Λ-graded algebra $\mathcal{B}$ has a multiloop realization based on a finite dimensional simple algebra if and only if $\mathcal{B}$ is graded-central-simple, $\mathcal{B}$ is a finitely generated module over its centroid $C(\mathcal{B})$, and the support $\Gamma(\mathcal{B})$ of $C(\mathcal{B})$ has finite index in Λ. This last condition on $\Gamma(\mathcal{B})$ is redundant in most cases of interest including Lie tori. Thus, returning to our original problem, it follows that a centreless Lie torus has a multiloop realization based on a finite dimensional simple algebra if and only if it is finitely generated as a module over its centroid.

We also obtain a graded-isomorphism theorem for two multiloop algebras based on a finite dimensional simple algebra. Roughly speaking the theorem states that two such multiloop algebras are graded-isomorphic, up to automorphism of the grading group, if and only if the sequences of commuting automorphisms that determine the algebras are in the same orbit under the natural action of $\mathrm{GL}_n(\mathbb{Z})$. (See Theorem 8.3.2(ii) for a more precise statement.)

For most of the paper, we work in a very general setting. We assume that Λ is an arbitrary abelian group, k is an arbitrary field, and study arbitrary graded-central-simple Λ-graded algebras over k. Rather than work with the multiloop algebra construction, which requires primitive roots of unity in k, we work instead with a more general loop construction. Given a fixed group epimorphism $\pi : \Lambda \to \bar{\Lambda}$ with kernel Γ, this general loop construction L_π produces a Λ-graded algebra $L_\pi(\mathcal{A})$ from a $\bar{\Lambda}$-graded algebra $\mathcal{A}$. This general point of view provides useful additional flexibility and simplicity even though our main interest is in multiloop algebras.

To study the construction L_π, it is convenient to introduce two classes of graded algebras: $\mathfrak{A}(\bar{\Lambda})$ is the class of all $\bar{\Lambda}$-graded algebras that are central-simple as algebras, whereas $\mathfrak{B}(\Lambda, \Gamma)$ is the class of all Λ-graded algebras $\mathcal{B}$ such that $\mathcal{B}$ is graded-central-simple, $\Gamma(\mathcal{B}) = \Gamma$, and the centroid $C(\mathcal{B})$ is split (isomorphic to the group algebra $k[\Gamma]$). If $\mathcal{A}$ is in $\mathfrak{A}(\bar{\Lambda})$, then $L_\pi(\mathcal{A})$ is in $\mathfrak{B}(\Lambda, \Gamma)$. Moreover the main result of the paper, which we call the Correspondence Theorem (Theorem 7.1.1), states that L_π establishes a 1-1 correspondence between similarity classes of $\bar{\Lambda}$-graded algebras in $\mathfrak{A}(\bar{\Lambda})$ and isomorphism classes of Λ-graded algebras in $\mathfrak{B}(\Lambda, \Gamma)$. (Similarity is discussed in §6.3.) To obtain the inverse of this correspondence we construct from any $\mathcal{B} \in \mathfrak{B}(\Lambda, \Gamma)$ the quotient algebra $\mathcal{B}/\ker(\rho)\mathcal{B}$ in $\mathfrak{A}(\bar{\Lambda})$, where $\rho : C(\mathcal{B}) \to k$ is an arbitrary unital algebra homomorphism. We call this quotient algebra a central image of $\mathcal{B}$. The freedom to choose ρ explains why we work with similarity classes in $\mathfrak{A}(\bar{\Lambda})$ rather than with graded-isomorphism classes.

All of the results mentioned above about multiloop algebras are obtained as consequences of the Correspondence Theorem.

To conclude this introduction, we briefly describe the contents of the paper. After a short preliminary section on graded algebras, we describe in Section 3 the general loop construction and the multiloop construction. In Section 4 we obtain some basic properties of the centroid and graded-central-simple algebras, and in Section 5 we look at those properties for loop algebras. In Section 6, we study central images and similarity.

Section 7 contains the Correspondence Theorem. To illustrate that there is interest in the case when k is not necessarily algebraically closed we include an example

of the correspondence in the associative case. In this example $\mathfrak{A}(\bar{\Lambda})$ contains many nonsplit finite dimensional central-simple algebras, all of which correspond to the same infinite dimensional algebra in $\mathfrak{B}(\Lambda, \Gamma)$, namely the quantum torus.

Section 8 applies the Correspondence Theorem to obtain our results about multiloop algebras. Finally, in Section 9 we discuss applications of our results to three classes of algebras that arise naturally as coordinate algebras in the study of EALA's. These are the associative, alternative and Jordan tori. We also briefly discuss the main application of this work to the study of Lie tori, but a detailed discussion of this application will be written in a sequel to this paper.

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2. Preliminaries

Throughout this work k denotes an arbitrary field. All algebras are assumed to be algebras (not necessarily associative or unital) over k. We also assume that Λ is an abelian group written additively.

In this section we recall some definitions and notation for Λ-graded algebras.

2.1. Definitions and notation.

Definition 2.1.1. A Λ-*graded algebra* is a pair $(\mathcal{B}, \Sigma)$ consisting of an algebra $\mathcal{B}$ together with a family $\Sigma = \{\, \mathcal{B}^\lambda \,\}_{\lambda\in\Lambda}$ of subspaces of $\mathcal{B}$ such that $\mathcal{B} = \bigoplus_{\lambda\in\Lambda} \mathcal{B}^\lambda$ and $\mathcal{B}^\lambda\mathcal{B}^\mu \subseteq \mathcal{B}^{\lambda+\mu}$ for $\lambda, \mu \in \Lambda$. We call Σ the Λ-*grading* of $(\mathcal{B}, \Sigma)$. We will usually suppress the symbol Σ in the notation and write the Λ-graded algebra $(\mathcal{B}, \Sigma)$ simply as $\mathcal{B}$.

Example 2.1.2. An important example of a Λ-graded algebra is the group algebra $k[\Lambda] = \bigoplus_{\lambda\in\Lambda} kz^\lambda$ of Λ, where the multiplication is given by $z^\lambda z^\mu = z^{\lambda+\mu}$ and the Λ-grading is given by $k[\Lambda]^\lambda = kz^\lambda$ for $\lambda \in \Lambda$.

Definition 2.1.3. There are two notions of isomorphism that we will use for graded algebras.

(i) Suppose that $\mathcal{B}$ and $\mathcal{B}'$ are Λ-graded algebras. We say that $\mathcal{B}$ and $\mathcal{B}'$ are *graded-isomorphic*, if there exists an algebra isomorphism $\varphi : \mathcal{B} \to \mathcal{B}'$ such that $\varphi(\mathcal{B}^\lambda) = \mathcal{B}'^\lambda$ for $\lambda \in \Lambda$. In that case we write $\mathcal{B} \simeq_\Lambda \mathcal{B}'$.

(ii) Suppose that $\mathcal{B}$ is a Λ-graded algebra and $\mathcal{B}'$ is a Λ'-graded algebra, where Λ' is another abelian group. We say that $\mathcal{B}$ and $\mathcal{B}'$ are *graded-isomorphic up to isomorphism of grading groups*, or more simply *isograded-isomorphic*, if there exists an algebra isomorphism $\varphi : \mathcal{B} \to \mathcal{B}'$ and a group isomorphism $\nu : \Lambda \to \Lambda'$ such that $\varphi(\mathcal{B}^\lambda) = \mathcal{B}'^{\nu(\lambda)}$ for $\lambda \in \Lambda$. In that case we write $\mathcal{B} \simeq_{\mathrm{ig}} \mathcal{B}'$.

Definition 2.1.4. Let $\mathcal{B}$ be a Λ-graded algebra and let $\nu \in \mathrm{Aut}(\Lambda)$. We may *regrade* $\mathcal{B}$ using ν to obtain a new Λ-graded algebra $\mathcal{B}_\nu$ as follows [17, §1.1]: As algebras $\mathcal{B}_\nu$ and $\mathcal{B}$ are the same, but the Λ-grading on $\mathcal{B}_\nu$ is defined by $(\mathcal{B}_\nu)^\lambda = \mathcal{B}^{\nu(\lambda)}$.

Remark 2.1.5. Suppose that $\mathcal{B}$ and $\mathcal{B}'$ are Λ-graded algebras. Then $\mathcal{B} \simeq_{\mathrm{ig}} \mathcal{B}'$ if and only if $\mathcal{B} \simeq_\Lambda \mathcal{B}'_\nu$ for some $\nu \in \mathrm{Aut}(\Lambda)$.

Notation 2.1.6. If $\mathcal{B} = \bigoplus_{\lambda\in\Lambda} \mathcal{B}^\lambda$ is a Λ-graded algebra over k, we use the notation

$$\mathrm{supp}_\Lambda(\mathcal{B}) := \{\, \lambda \in \Lambda \mid \mathcal{B}^\lambda \neq 0 \,\}$$

for the Λ-*support* of $\mathcal{B}$. We denote the subgroup of Λ generated by $\operatorname{supp}_\Lambda(\mathcal{B})$ as $\langle \operatorname{supp}_\Lambda(\mathcal{B})\rangle$.

3. Loop algebras and multiloop algebras

In this section we introduce the main object of our study—the loop algebra $L_\pi(\mathcal{A})$. The definition (and many of the results of this paper) assumes only that Λ is an abelian group.

3.1. **The general definition.**

The following definition is given in the associative case in [18, Proposition 1.2.2].

Definition 3.1.1. Suppose that $\pi : \Lambda \to \bar{\Lambda}$ is a group epimorphism of an abelian group Λ onto an abelian group $\bar{\Lambda}$. We write

$$\bar{\lambda} = \pi(\lambda)$$

for $\lambda \in \Lambda$. Suppose that $\mathcal{A} = \bigoplus_{\bar{\lambda}\in\bar{\Lambda}} \mathcal{A}^{\bar{\lambda}}$ is a $\bar{\Lambda}$-graded algebra. Then the tensor product $\mathcal{A} \otimes_k k[\Lambda]$ is a Λ-graded algebra over k, where $(\mathcal{A} \otimes_k k[\Lambda])^\lambda = \mathcal{A} \otimes z^\lambda$ for $\lambda \in \Lambda$. We define

$$L_\pi(\mathcal{A}) = \sum_{\lambda\in\Lambda} \mathcal{A}^{\bar{\lambda}} \otimes z^\lambda$$

in $\mathcal{A} \otimes_k k[\Lambda]$. Then $L_\pi(\mathcal{A})$ is a Λ-graded subalgebra of $\mathcal{A} \otimes_k k[\Lambda]$. Hence $L_\pi(\mathcal{A})$ is a Λ-graded algebra with

$$L_\pi(\mathcal{A})^\lambda = \mathcal{A}^{\bar{\lambda}} \otimes z^\lambda$$

for $\lambda \in \Lambda$. We call $L_\pi(\mathcal{A})$ the *loop algebra* of $\mathcal{A}$ relative to the π. If we wish to emphasize the role of the grading $\Sigma = \{\mathcal{A}^{\bar{\lambda}}\}_{\bar{\lambda}\in\bar{\Lambda}}$ of $\mathcal{A}$ in the loop construction, we write $L_\pi(\mathcal{A})$ as $L_\pi(\mathcal{A}, \Sigma)$.

Remark 3.1.2. Let $\pi : \Lambda \to \bar{\Lambda}$ be an epimorphism.

(i) L_π is a functor from the category of $\bar{\Lambda}$-graded algebras to the category of Λ-graded algebras. (The morphisms in each case are the graded homomorphisms.)

(ii) If $\mathcal{A}$ is a $\bar{\Lambda}$-graded algebra, then there is a unique linear map from $L_\pi(\mathcal{A})$ to $\mathcal{A}$ such that

$$u \otimes z^\lambda \mapsto u$$

for $\lambda \in \Lambda$ and $u \in \mathcal{A}^{\bar{\lambda}}$. This map is an (ungraded) algebra epimorphism of $L_\pi(\mathcal{A})$ onto $\mathcal{A}$, and consequently the algebra $\mathcal{A}$ is a homomorphic image of the algebra $L_\pi(\mathcal{A})$. This fact will be very important later in this work (see Proposition 6.5.2).

(iii) It is clear that $\mathcal{A}$ is a Lie algebra if and only if $L_\pi(\mathcal{A})$ is a Lie algebra. Thus a reader whose primary interest is in Lie algebras can chose to assume throughout this paper that all algebras discussed are Lie algebras. A similar remark can be made for associative algebras, or alternative algebras or Jordan algebras (if k has characteristic $\neq 2$).

3.2. **Multiloop algebras.**

In this subsection we assume that $\Lambda = \mathbb{Z}^n$ and consider multiloop algebras graded by $\mathbb{Z}^n$. These are a special case of the loop algebras just described.

If $\ell \geq 1$, an element $\zeta_\ell \in k^\times$ is called (as usual) a *primitive ℓ^{th}-root of unity in k* if ζ_ℓ generates a subgroup of $k^\times$ of order ℓ.

Definition 3.2.1. Let $\Lambda = \mathbb{Z}^n$. Suppose that $\mathbf{m} = (m_1, \dots, m_n)$ is an n-tuple of positive integers such that k contains an primitive ℓ^{th} root of unity ζ_ℓ (which we fix) for $\ell \in \{ m_1, \dots, m_n \}$. To construct a multiloop algebra, suppose that $\mathcal{A}$ is an (ungraded) algebra over k and suppose that $\sigma_1, \dots, \sigma_n$ is a sequence of (pairwise) commuting finite order algebra automorphisms of $\mathcal{A}$ such that $\sigma_i^{m_i} = 1$ for each i. Set

$$\bar{\Lambda} = \mathbb{Z}/(m_1) \oplus \cdots \oplus \mathbb{Z}/(m_n),$$

Then $\mathcal{A}$ has the $\bar{\Lambda}$-grading $\Sigma = \{ \mathcal{A}^{\bar{\lambda}} \}_{\bar{\lambda} \in \bar{\Lambda}}$ defined by

$$\mathcal{A}^{(\bar{\ell}_1, \dots, \bar{\ell}_n)} = \{ u \in \mathcal{A} \mid \sigma_j u = \zeta_{m_j}^{\ell_j} u \text{ for } 1 \le j \le n \} \tag{1}$$

for $\ell_1, \dots, \ell_n \in \mathbb{Z}$, where $\bar{\ell}_j = \ell_j + m_j\mathbb{Z}$ for each j. We call Σ the *$\bar{\Lambda}$-grading of $\mathcal{A}$ determined by the automorphisms* $\sigma_1, \dots, \sigma_n$. Let $k[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ be the algebra of Laurent polynomials over k and let

$$M_{\mathbf{m}}(\mathcal{A}, \sigma_1, \dots, \sigma_n) = \sum_{(\ell_1, \dots, \ell_n) \in \mathbb{Z}^n} \mathcal{A}^{(\bar{\ell}_1, \dots, \bar{\ell}_n)} \otimes z_1^{\ell_1} \cdots z_n^{\ell_n} \subseteq \mathcal{A} \otimes_k k[z_1^{\pm 1}, \dots, z_n^{\pm 1}].$$

Then $M_{\mathbf{m}}(\mathcal{A}, \sigma_1, \dots, \sigma_n)$ is a subalgebra of $\mathcal{A} \otimes_k k[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$. We define a Λ-grading on $M_{\mathbf{m}}(\mathcal{A}, \sigma_1, \dots, \sigma_n)$ by setting

$$M_{\mathbf{m}}(\mathcal{A}, \sigma_1, \dots, \sigma_n)^{(\ell_1, \dots, \ell_n)} = \mathcal{A}^{(\bar{\ell}_1, \dots, \bar{\ell}_n)} \otimes z_1^{\ell_1} \cdots z_n^{\ell_n} \tag{2}$$

for all $(\ell_1, \dots, \ell_n) \in \mathbb{Z}^n$. We call the $\mathbb{Z}^n$-graded algebra $M_{\mathbf{m}}(\mathcal{A}, \sigma_1, \dots, \sigma_n)$ the *multiloop algebra* of $\sigma_1, \dots, \sigma_n$ (based on $\mathcal{A}$ and relative to $\mathbf{m}$).

It is clear that this multiloop algebra construction is a special case of the general loop algebra construction described in Definition 3.1.1. Indeed, let $\pi : \Lambda \to \bar{\Lambda}$ be the natural map defined by

$$\pi(\ell_1, \dots, \ell_n) = \overline{(\ell_1, \dots, \ell_n)} := (\bar{\ell}_1, \dots, \bar{\ell}_n), \tag{3}$$

for $(\ell_1, \dots, \ell_n) \in \Lambda$, and identify $k[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ with $k[\Lambda]$ by means of the map $z_1^{\ell_1} \dots z_n^{\ell_n} \mapsto z^{(\ell_1, \dots, \ell_n)}$. Then we have

$$M_{\mathbf{m}}(\mathcal{A}, \sigma_1, \dots, \sigma_n) = L_\pi(\mathcal{A}, \Sigma). \tag{4}$$

(To avoid confusion we are not abbreviating the graded algebra $(\mathcal{A}, \Sigma)$ as $\mathcal{A}$ here.)

Remark 3.2.2. Our main interest is in multiloop algebras. However, as we'll see in the rest of this work, the coordinate free point of view in the general loop construction provides us with valuable flexibility.

Remark 3.2.3. Assume that Λ, $\mathbf{m}$, $\mathcal{A}$, $\sigma_1, \dots, \sigma_n$ and $\bar{\Lambda}$ are as in Definition 3.2.1.

(i) If $n = 1$, the $\mathbb{Z}$-graded algebra $M_{\mathbf{m}}(\mathcal{A}, \sigma_1)$ is known classically as the loop algebra of the automorphism σ_1 [14, §8.2].

(ii) Although we have suppressed this from the notation (for simplicity), the multiloop algebra $M_{\mathbf{m}}(\mathcal{A}, \sigma_1, \dots, \sigma_n)$ does depend on the choice of the roots of unity ζ_ℓ, $\ell \in \{ m_1, \dots, m_n \}$.

(iii) There is an alternate way to view the $\bar{\Lambda}$-grading on $\mathcal{A}$ determined by $\sigma_1, \dots, \sigma_n$. To describe this, let $G = \langle \sigma_1, \dots, \sigma_n \rangle$ and let $\widehat{G} = \operatorname{Hom}(G, k^\times)$ be the character group of G. We write

$$\sigma^{\boldsymbol{\ell}} = \prod_{i=1}^n \sigma_i^{l_i} \in G$$

for $\boldsymbol{\ell} = (l_1, \ldots, l_n) \in \Lambda$. Then the map $\boldsymbol{\ell} \mapsto \sigma^{\boldsymbol{\ell}}$ is a group epimomorphism of Λ onto G that induces a group epimorphism $\eta : \bar{\Lambda} \to G$ given by

$$\eta(\bar{\boldsymbol{\ell}}) = \sigma^{\boldsymbol{\ell}} \tag{5}$$

for $\boldsymbol{\ell} \in \Lambda$. So η determines a group monomorphism $\hat{\eta} : \widehat{G} \to \widehat{\bar{\Lambda}}$ with $\hat{\eta}(\chi) = \chi \circ \eta$. Next the choice of the roots of unity ζ_ℓ, $\ell \in \{ m_1, \ldots, m_n \}$, defines a nondegenerate pairing $\bar{\Lambda} \times \bar{\Lambda} \to k^\times$ with

$$(\bar{\mathbf{k}}, \bar{\boldsymbol{\ell}}) \to \langle \bar{\mathbf{k}}, \bar{\boldsymbol{\ell}} \rangle = \textstyle\prod_{i=1}^n \zeta_{m_i}^{k_i \ell_i}.$$

for $\bar{\mathbf{k}} = (\bar{k}_1, \ldots, \bar{k}_n)$ and $\bar{\boldsymbol{\ell}} = (\bar{\ell}_1, \ldots, \bar{\ell}_n)$ in $\bar{\Lambda}$. This pairing gives an isomorphism $\psi : \bar{\Lambda} \to \widehat{\bar{\Lambda}}$ with $\psi(\bar{\boldsymbol{\ell}}) = \langle \bar{\boldsymbol{\ell}}, \ \rangle$ [16, §1.9, Theorem 9.2]. Consequently

$$\psi^{-1} \circ \hat{\eta} : \widehat{G} \to \bar{\Lambda}$$

is a group monomorphism. But $\mathcal{A}$ is naturally a $\widehat{G}$-graded algebra (although $\widehat{G}$ is written multiplicatively) with $\mathcal{A}^\chi = \{a \in \mathcal{A} \mid g(a) = \chi(g)a \text{ for all } g \in G\}$ [18, Remark 1.3.14]. The monomorphism $\psi^{-1} \circ \hat{\eta}$ transfers the $\widehat{G}$-grading of $\mathcal{A}$ to a $\bar{\Lambda}$-grading. This transferred grading coincides with the $\bar{\Lambda}$-grading determined by $\sigma_1, \ldots, \sigma_n$. Indeed, if $\bar{\boldsymbol{\ell}} = (\psi^{-1} \circ \hat{\eta})(\chi)$, where $\chi \in \widehat{G}$, then $\psi(\bar{\boldsymbol{\ell}}) = \chi \circ \eta$ and so

$$\chi(\sigma_i) = (\chi \circ \eta)(\mathbf{e}_i) = \langle \bar{\boldsymbol{\ell}}, \bar{\mathbf{e}}_i \rangle = \zeta_{m_i}^{\ell_i}$$

where $\mathbf{e}_i$ is the i^{th} standard basis vector of Λ. Thus, $\mathcal{A}^\chi = \mathcal{A}^{\bar{\boldsymbol{\ell}}}$.

It is sometimes convenient to work with Λ-graded algebras $\mathcal{B}$ that satisfy the condition $\langle \operatorname{supp}_\Lambda(\mathcal{B}) \rangle = \Lambda$. For example this is done in the study of various classes of tori (see §9). Therefore, the following lemma will be useful.

Lemma 3.2.4. *Let Λ, $\mathbf{m}$, $\mathcal{A}$, $\sigma_1, \ldots, \sigma_n$, $\bar{\Lambda}$ and $\mathcal{B} = M_{\mathbf{m}}(\mathcal{A}, \sigma_1, \ldots, \sigma_n)$ be as in Definition 3.2.1 and let $\eta : \bar{\Lambda} \to G = \langle \sigma_1, \ldots, \sigma_n \rangle$ be the group epimorphism defined by* (5). *Then the following are equivalent:*

(a) $\langle \operatorname{supp}_\Lambda(\mathcal{B}) \rangle = \Lambda$,
(b) $|G| = m_1 \cdots m_n$
(c) *η is an isomorphism.*

Proof. Clearly, (b) and (c) are equivalent. We show that (a) and (c) are equivalent. It is clear that

$$\langle \operatorname{supp}_\Lambda(\mathcal{B}) \rangle = \Lambda \iff \langle \operatorname{supp}_{\bar{\Lambda}}(\mathcal{A}) \rangle = \bar{\Lambda}.$$

Using the $\widehat{G}$-grading of $\mathcal{A}$ as in Remark 3.2.3(iii), let $H = \langle \operatorname{supp}_{\widehat{G}}(\mathcal{A}) \rangle$. Now the natural pairing $H \times G \to k^\times$ is nondegenerate. Indeed, the left kernel is trivial since H consists of functions on G and the right kernel is trivial since G acts faithfully on $\mathcal{A}$. Thus, $H = \widehat{G}$ [16, §1.9, Theorem 9.2]. So $\langle \operatorname{supp}_{\widehat{G}}(\mathcal{A}) \rangle = \widehat{G}$ and hence, by Remark 3.2.3(iii),

$$\langle \operatorname{supp}_{\bar{\Lambda}}(\mathcal{A}) \rangle = \psi^{-1}(\hat{\eta}(\widehat{G})).$$

Therefore

$$\langle \operatorname{supp}_{\bar{\Lambda}}(\mathcal{A}) \rangle = \bar{\Lambda} \iff \hat{\eta}(\widehat{G}) = \widehat{\bar{\Lambda}} \iff \hat{\eta} \text{ is an isomorphism.}$$

But, since G and $\bar{\Lambda}$ are finite, $\hat{\eta}$ is an isomorphism if and only if η is an isomorphism. $\square$

4. Graded-central-simple algebras

We assume again that Λ is an arbitrary abelian group. In preparation for our results on the realization of graded-central-simple algebras, we discuss in this section some of the basic properties of these algebras.

4.1. **The centroid.**

Definition 4.1.1. Suppose that $\mathcal{B}$ is an algebra. Let $\mathrm{Mult}_k(\mathcal{B})$ be the unital subalgebra of $\mathrm{End}_k(\mathcal{B})$ generated by $\{1\} \cup \{l_a \mid a \in \mathcal{B}\} \cup \{r_a \mid a \in \mathcal{B}\}$, where l_a (resp. r_a) denotes the left (resp. right) multiplication operator by a. $\mathrm{Mult}_k(\mathcal{B})$ is called the *multiplication algebra* of $\mathcal{B}$. Let $C_k(\mathcal{B})$ denote the centralizer of $\mathrm{Mult}_k(\mathcal{B})$ in $\mathrm{End}_k(\mathcal{B})$. Then $C_k(\mathcal{B})$ is a unital subalgebra of $\mathrm{End}_k(\mathcal{B})$ called the *centroid* of $\mathcal{B}$.

From now on we will usually abbreviate $\mathrm{Mult}_k(\mathcal{B})$ and $C_k(\mathcal{B})$ as $\mathrm{Mult}(\mathcal{B})$ and $C(\mathcal{B})$ respectively.

Remark 4.1.2. If $\mathcal{B}$ is a unital algebra, then the map $a \mapsto l_a$ is an algebra isomorphism of the centre of $\mathcal{B}$ onto $C(\mathcal{B})$. (See for example [12, §1].)

Suppose that $\mathcal{B} = \bigoplus_{\lambda\in\Lambda} \mathcal{B}^\lambda$ is a Λ-graded algebra. For $\lambda \in \Lambda$, we let

$$\mathrm{End}_k(\mathcal{B})^\lambda = \{ e \in \mathrm{End}_k(\mathcal{B}) \mid e(\mathcal{B}^\mu) \subseteq \mathcal{B}^{\lambda+\mu} \text{ for } \mu \in \Lambda \}.$$

Then $\bigoplus_{\lambda\in\Lambda} \mathrm{End}_k(\mathcal{B})^\lambda$ is subalgebra of $\mathrm{End}_k(\mathcal{B})$ that is Λ-graded. We set

$$\mathrm{Mult}(\mathcal{B})^\lambda = \mathrm{Mult}(\mathcal{B}) \cap \mathrm{End}_k(\mathcal{B})^\lambda \quad \text{and} \quad C(\mathcal{B})^\lambda = C(\mathcal{B}) \cap \mathrm{End}_k(\mathcal{B})^\lambda \tag{6}$$

for $\lambda \in \Lambda$. It is clear that $\mathrm{Mult}(\mathcal{B}) = \bigoplus_{\lambda\in\Lambda} \mathrm{Mult}(\mathcal{B})^\lambda$, and hence $\mathrm{Mult}(\mathcal{B})$ is Λ-graded. Although the centroid is not in general Λ-graded, it does have this property in many important cases (see for example Lemma 4.2.3 below).

4.2. **Graded-simplicity.**

Definition 4.2.1. If $\mathcal{B}$ is an algebra we say (as is usual) that $\mathcal{B}$ is *simple* if $\mathcal{B}\mathcal{B} \neq 0$ and the only ideals of $\mathcal{B}$ are 0 and $\mathcal{B}$. If $\mathcal{B}$ is a Λ-graded algebra we say that $\mathcal{B}$ is *graded-simple* (or simple as a graded algebra) if $\mathcal{B}\mathcal{B} \neq 0$ and the only graded ideals of $\mathcal{B}$ are 0 and $\mathcal{B}$.

Clearly if $\mathcal{B}$ is a Λ-graded algebra and $\mathcal{B}\mathcal{B} \neq 0$ then

$$\mathcal{B} \text{ is graded-simple} \iff \begin{array}{l}\text{For each nonzero homogeneous element}\\ x \in \mathcal{B} \text{ we have } \mathcal{B} = \mathrm{Mult}(\mathcal{B})x.\end{array} \tag{7}$$

Lemma 4.2.2. *Suppose that $\mathcal{B}$ is a Λ-graded algebra. Then*

$$\mathcal{B} \textit{ is simple} \iff \mathcal{B} \textit{ is graded-simple and } C(\mathcal{B}) \textit{ is a field.} \tag{8}$$

Consequently, if $C(\mathcal{B}) = k1$ and $\mathcal{B}$ is graded-simple then $\mathcal{B}$ is simple.

Proof. It is enough to prove (8). The implication "$\Rightarrow$" is clear (and well-known). For the proof of "$\Leftarrow$", suppose that $\mathcal{B}$ is graded-simple and $C(\mathcal{B})$ is a field. Let $\mathcal{J}$ be a nonzero ideal of $\mathcal{B}$. Choose a nonzero element

$$x = \sum_{i=1}^{\ell} x_i$$

in $\mathcal{J}$, where $0 \neq x_i \in \mathcal{B}^{\lambda_i}$ for all i and $\lambda_i \neq \lambda_j$ for $i \neq j$. We assume that x is chosen such that ℓ is minimum. Since $\mathcal{B}$ is graded-simple, we have

$$\mathcal{B} = \mathrm{Mult}(\mathcal{B})x_1. \tag{9}$$

If $\ell = 1$ then $\mathrm{Mult}(\mathcal{B})x = \mathcal{B}$ and so $\mathcal{I} = \mathcal{B}$. So we can assume that $\ell \geq 2$.

If m is a homogeneous element of $\mathrm{Mult}(\mathcal{B})$ and $1 \leq i \leq \ell$, then $mx_1 = 0$ if and only if $mx_i = 0$ (by the minimality of ℓ). Consequently if m and n are homogeneous elements of $\mathrm{Mult}(\mathcal{B})$ of the same degree and $1 \leq i \leq \ell$, then

$$mx_1 = nx_1 \iff mx_i = nx_i. \tag{10}$$

Let $1 \leq i \leq \ell$. Then, by (9) and (10), there exists a well-defined map $c_i \in \mathrm{End}_k(\mathcal{B})^{\lambda_i - \lambda_1}$ such that

$$c_i(mx_1) = mx_i$$

for any homogeneous $m \in \mathrm{Mult}(\mathcal{B})$. In particular

$$c_i x_1 = x_i.$$

Observe also that if m and n are homogeneous elements of $\mathrm{Mult}(\mathcal{B})$ then $c_i(mnx_1) = mnx_i = mc_i(nx_1)$. Hence $c_i \in C(\mathcal{B})^{\lambda_i - \lambda_1}$.

We now put $c = \sum_{i=1}^{\ell} c_i$ in which case $c \in C(\mathcal{B})$ and $x = cx_1$. Thus, since c is invertible, we have

$$\mathcal{B} = c\mathcal{B} = c\,\mathrm{Mult}(\mathcal{B})x_1 = \mathrm{Mult}(\mathcal{B})cx_1 = \mathrm{Mult}(\mathcal{B})x \subseteq \mathcal{I},$$

and hence $\mathcal{I} = \mathcal{B}$. □

The following is proved in [5, Proposition 2.16]:

Lemma 4.2.3. *Suppose that $\mathcal{B}$ is a graded-simple Λ-graded algebra. Then*

(i) $\mathcal{B} = \mathcal{B}\mathcal{B}$ *and so* $C(\mathcal{B})$ *is commutative.*
(ii) $C(\mathcal{B}) = \bigoplus_{\lambda \in \Lambda} C(\mathcal{B})^{\lambda}$, *and so* $C(\mathcal{B})$ *is a* Λ*-graded algebra.*
(iii) *Each nonzero homogeneous element of* $C(\mathcal{B})$ *is invertible in* $C(\mathcal{B})$.
(iv) $C(\mathcal{B})^0$ *is a field.*
(v) $\mathcal{B}$ *and* $C(\mathcal{B})$ *are naturally* Λ*-graded algebras over the field* $C(\mathcal{B})^0$.

Definition 4.2.4. If $\mathcal{B}$ is a graded-simple Λ-graded algebra, we put

$$\Gamma_\Lambda(\mathcal{B}) := \mathrm{supp}_\Lambda(C(\mathcal{B})) = \{\, \gamma \in \Lambda \mid C(\mathcal{B})^{\gamma} \neq 0 \,\}.$$

$\Gamma_\Lambda(\mathcal{B})$ is a subgroup of Λ by Lemma 4.2.3(iii). We call $\Gamma_\Lambda(\mathcal{B})$ the *central grading group* of $\mathcal{B}$. ($\Gamma_\Lambda(\mathcal{B})$ is also called the centroid grading group of $\mathcal{B}$ [20, §6].) From now on we will usually abbreviate $\Gamma_\Lambda(\mathcal{B})$ as $\Gamma(\mathcal{B})$.

Remark 4.2.5. Suppose that $\mathcal{B}$ is a graded-simple Λ-graded algebra. If $\mathcal{B}_\nu$ is obtained from $\mathcal{B}$ by regrading using $\nu \in \mathrm{Aut}(\Lambda)$ (see Definition 2.1.4), then $\Gamma(\mathcal{B}_\nu) = \nu^{-1}(\Gamma(\mathcal{B}))$.

4.3. **Graded-centrality.**

Definition 4.3.1. Suppose that $\mathcal{B}$ is an algebra. Then, $k1 \subseteq C(\mathcal{B})$, and we say that $\mathcal{B}$ is *central* if $C(\mathcal{B}) = k1$. We say that $\mathcal{B}$ is *central-simple* if $\mathcal{B}$ is central and simple. Recall that if k is algebraically closed, then any finite dimensional simple algebra is automatically central-simple [13, Theorem 10.1].

Suppose next that $\mathcal{B}$ is a Λ-graded algebra. Then $k1 \subseteq C(\mathcal{B})^0 \subseteq C(\mathcal{B})$, where $C(\mathcal{B})^0 = \{\, c \in C(\mathcal{B}) \mid c(\mathcal{B}^\lambda) \subseteq \mathcal{B}^\lambda \text{ for } \lambda \in \Lambda \,\}$ (see (6)). We say that $\mathcal{B}$ is *graded-central* if $C(\mathcal{B})^0 = k1$. Further, we say that $\mathcal{B}$ is *graded-central-simple* if $\mathcal{B}$ is graded-central and $\mathcal{B}$ is graded-simple.

Remark 4.3.2. There are some basic properties of graded-central-simple algebras that suggest that they are the natural analogs of central-simple algebras in the ungraded theory (see for example [13, Section 1, Chapter X]). We describe these properties here, omitting proofs since we will not make use of the properties in this paper.

(i) If $\mathcal{B}$ is a graded-simple Λ-graded k-algebra and $K = C(\mathcal{B})^0$, then K/k is a field extension (see parts (iv) and (v) in Lemma 4.2.3), and $\mathcal{B}$ is naturally a graded-central-simple Λ-graded K-algebra. Conversely, if K/k is a field extension and $\mathcal{B}$ is a graded-central-simple Λ-graded K-algebra, one can easily show that $\mathcal{B}$ is a graded-simple Λ-graded k-algebra and $C(\mathcal{B})^0 = K1$.

(ii) Suppose that $\mathcal{B}$ is a graded-central-simple algebra over k and K/k is a field extension. Then one can show that $\mathcal{B} \otimes_k K$ is a graded-central-simple Λ-graded algebra over K.

Remark 4.3.3. When $\Lambda = \mathbb{Z}/2\mathbb{Z}$, finite dimensional associative unital graded-central-simple Λ-graded algebras have been classified by C.T.C. Wall [24] and they play an important role in the theory of quadratic forms [15, Chapters IV and V].

There are two cases when a graded-simple Λ-graded algebra is automatically graded-central.

Lemma 4.3.4. *Let $\mathcal{B}$ be a graded-simple Λ-graded algebra. Suppose either that $\dim \mathcal{B}^\lambda = 1$ for some $\lambda \in \Lambda$ or that k is algebraically closed and $0 < \dim \mathcal{B}^\lambda < \infty$ for some $\lambda \in \Lambda$. Then $\mathcal{B}$ is graded-central-simple.*

Proof. Choose $0 \neq x \in \mathcal{B}^\lambda$. Since $C(\mathcal{B})^0$ is a field by Lemma 4.2.3(iv), the map $c \mapsto cx$ is a linear injection of $C(\mathcal{B})^0$ into $\mathcal{B}^\lambda$. So $\dim C(\mathcal{B})^0 \leq \dim \mathcal{B}^\lambda$. If $\dim \mathcal{B}^\lambda = 1$ then $C(\mathcal{B})^0 = k1$. On the other hand if k is algebraically closed and $\dim \mathcal{B}^\lambda < \infty$, then $C(\mathcal{B})^0/k1$ is a finite extension and so again $C(\mathcal{B})^0 = k1$. □

In view of Remark 4.3.2(i), the study of graded-simple algebras over k can be regarded as equivalent to the study of graded-central-simple algebras over extensions of k. With this in mind, we concentrate in the rest of this paper on the study of graded-central-simple algebras.

We first look at the structure of the centroid. The next lemma tells us that $C(\mathcal{B})$ is a twisted group algebra of $\Gamma(\mathcal{B})$ over k [22, §1.2].

Lemma 4.3.5. *Suppose that $\mathcal{B}$ is a graded-central-simple Λ-graded algebra. Then $C(\mathcal{B})$ has a basis $\{ c_\gamma \}_{\gamma \in \Gamma(\mathcal{B})}$ such that $c_\gamma \in C(\mathcal{B})^\gamma$ is a unit of $C(\mathcal{B})$ for $\gamma \in \Gamma(\mathcal{B})$. Hence if $\gamma \in \Gamma(\mathcal{B})$ and $\lambda \in \Lambda$, then $\mathcal{B}^{\gamma+\lambda} = c_\gamma \mathcal{B}^\lambda$.*

Proof. As observed in [5, §2.2], this follows from Lemma 4.2.3 and the fact that $C(\mathcal{B})^0 = k1$. □

Definition 4.3.6. Suppose that $\mathcal{B}$ is a graded-central-simple Λ-graded algebra. We say that the centroid $C(\mathcal{B})$ of $\mathcal{B}$ is *split* if

$$C(\mathcal{B}) \simeq_{\Gamma(\mathcal{B})} k[\Gamma(\mathcal{B})]. \tag{11}$$

Note that both $C(\mathcal{B})$ and $k[\Gamma(\mathcal{B})]$ are Λ-graded since $\Gamma(\mathcal{B})$ is a subgroup of Λ. Thus (11) can alternately be written as $C(\mathcal{B}) \simeq_\Lambda k[\Gamma(\mathcal{B})]$. Note also that $C(\mathcal{B})$ is split if and only a basis $\{ c_\gamma \}_{\gamma \in \Gamma}$ for $C(\mathcal{B})$ can be chosen as in Lemma 4.3.5 with the additional property that

$$c_\gamma c_\delta = c_{\gamma+\delta} \tag{12}$$

for $\gamma, \delta \in \Gamma$.

If $\mathcal{B}$ is an algebra, let

$$\mathrm{Alg}(C(\mathcal{B}), k)$$

denote the set of all unital k-algebra homomorphisms of $C(\mathcal{B})$ into k.

Lemma 4.3.7. *Suppose that $\mathcal{B}$ is a graded-central-simple Λ-graded algebra. Then*

$$C(\mathcal{B}) \textit{ is split } \iff \mathrm{Alg}(C(\mathcal{B}), k) \neq \emptyset.$$

Proof. Since $C(\mathcal{B})$ is a twisted group ring, this is Exercise 17 in Chapter 1 of [22]. We include a proof for the reader's convenience.

"$\Rightarrow$" For this implication we can assume that the elements c_γ in Lemma 4.3.5 satisfy (12). Then the augmentation map $c_\gamma \mapsto 1$ is an element of $\mathrm{Alg}(C(\mathcal{B}), k)$.

"$\Leftarrow$" Suppose that $\rho \in \mathrm{Alg}(C(\mathcal{B}), k)$. Choose c_γ, $\gamma \in \Gamma(\mathcal{B})$, as in Lemma 4.3.5. Then $\rho(c_\gamma)$ is a unit in k and we set

$$d_\gamma = \rho(c_\gamma)^{-1} c_\gamma \in C(\mathcal{B})^\gamma$$

for $\gamma \in \Gamma(\mathcal{B})$. Now for $\gamma, \delta \in \Gamma(\mathcal{B})$, we have

$$d_\gamma d_\delta = \rho(c_\gamma)^{-1} c_\gamma \rho(c_\delta)^{-1} c_\delta = \rho(c_\gamma c_\delta)^{-1} c_\gamma c_\delta = \rho(c_{\gamma+\delta})^{-1} c_{\gamma+\delta},$$

where the last equality holds since $c_\gamma c_\delta$ is a nonzero scalar multiple of $c_{\gamma+\delta}$. Hence $d_\gamma d_\delta = d_{\gamma+\delta}$ as needed. □

There are two cases when $C(\mathcal{B})$ is always split.

Lemma 4.3.8. *Suppose that Λ is finitely generated and free, or that k is algebraically closed. If $\mathcal{B}$ is a graded-central-simple Λ-graded algebra, then the centroid of $\mathcal{B}$ is split.*

Proof. If Λ is finitely generated and free, then so in $\Gamma(\mathcal{B})$ and hence (11) is clear using Lemma 4.3.5. Assume next that k is algebraically closed. Then $C(\mathcal{B})$ is a commutative twisted group algebra of an abelian group over an algebraically closed field and so (11) holds by [22, Lemma 1.2.9(i)]. □

4.4. **Fgc graded-central-simple algebras.** If $\mathcal{B}$ is an algebra, then $\mathcal{B}$ is a (left) module over its centroid $C(\mathcal{B})$. We now look at this structure.

The following lemma follows from Lemma 4.3.5 and the fact that $\Gamma(\mathcal{B})$ acts freely on Λ (that is $\gamma + \lambda = \lambda$ implies $\gamma = 0$ for $\gamma \in \Gamma(\mathcal{B})$ and $\lambda \in \Lambda$). (See [8, Theorem 3] or [21, Lemma 2.8(ii)]).

Lemma 4.4.1. *Suppose that $\mathcal{B}$ is a graded-central-simple Λ-graded algebra. Choose a set Θ of coset representatives of $\Gamma(\mathcal{B})$ in Λ, and for $\theta \in \Theta$, choose a k-basis X^θ for $\mathcal{B}^\theta$. Using these choices let*

$$X = \cup_{\theta \in \Theta} X^\theta.$$

Then X is a homogeneous $C(\mathcal{B})$-basis for $\mathcal{B}$. Hence $\mathcal{B}$ is a free $C(\mathcal{B})$-module of rank $\sum_{\theta \in \Theta} \dim_k(\mathcal{B}^\theta)$ (where we interpret the sum on the right as ∞ if any of the terms in the sum is infinite or if there are infinitely many nonzero terms in the sum).

Remark 4.4.2. More generally, suppose $\mathcal{B}$ is a graded-central-simple Λ-graded algebra and $\mathcal{M}$ is a Λ-graded $C(\mathcal{B})$-submodule of $\mathcal{B}$. Then (for the reasons mentioned before the statement of Lemma 4.4.1) $\mathcal{M}$ has a homogeneous $C(\mathcal{B})$-basis and $\mathcal{M}$ is a free $C(\mathcal{B})$-module of rank $\sum_{\theta \in \Theta} \dim_k(\mathcal{M}^\theta)$, where Θ is as in Lemma 4.4.1.

Notation 4.4.3. Suppose that $\mathcal{B}$ is a graded-central-simple Λ-graded algebra. By the last statement in Lemma 4.3.5, we see that $\operatorname{supp}_\Lambda(\mathcal{B})$ is the union of cosets of $\Gamma(\mathcal{B})$ in Λ. We let

$$\operatorname{supp}_\Lambda(\mathcal{B})/\Gamma(\mathcal{B})$$

denote the set of all cosets of $\Gamma(\mathcal{B})$ in Λ that are represented by elements of $\operatorname{supp}_\Lambda(\mathcal{B})$ (and hence consist entirely of elements of $\operatorname{supp}_\Lambda(\mathcal{B})$).

Definition 4.4.4. If $\mathcal{B}$ is an algebra we say that $\mathcal{B}$ is *fgc* if $\mathcal{B}$ is finitely generated as a $C(\mathcal{B})$-module. (The term fgc is of course an acronym for <u>f</u>initely <u>g</u>enerated as a module over its <u>c</u>entroid.)

Proposition 4.4.5. *If $\mathcal{B}$ is a graded-central-simple Λ-graded algebra, then the following are equivalent:*

(a) $\mathcal{B}$ *is fgc.*
(b) $\mathcal{B}$ *is a free module of finite rank over* $C(\mathcal{B})$*.*
(c) $\operatorname{supp}_\Lambda(\mathcal{B})/\Gamma(\mathcal{B})$ *is finite and* $\dim(\mathcal{B}^\lambda) < \infty$ *for all* $\lambda \in \Lambda$*.*

Also (a), (b) and (c) are implied by

(d) $\Lambda/\Gamma(\mathcal{B})$ *is finite and* $\dim(\mathcal{B}^\lambda) < \infty$ *for all* $\lambda \in \Lambda$*.*

Moreover, if Λ is finitely generated and $\ell\Lambda \subseteq \operatorname{supp}_\Lambda(\mathcal{B})$ for some positive integer ℓ, then (a), (b), (c) and (d) are equivalent.

Proof. The equivalence of (a), (b) and (c) follow from Lemma 4.4.1, and the implication "(d) $\Rightarrow$ (c)" is trivial. It remains to show that (c) implies (d) when the additional assumptions on Λ and $\operatorname{supp}_\Lambda(\mathcal{B})$ hold. Let $\bar\Lambda = \Lambda/\Gamma(\mathcal{B})$ and let $\bar{} : \Lambda \to \bar\Lambda$ be the canonical projection. Set $T = \operatorname{supp}_\Lambda(\mathcal{B})$. Then, since (c) holds, $\bar T$ is finite. But $\ell\bar\Lambda \subseteq \bar T$ and so $\ell\bar\Lambda$ is finite. On the other hand, since Λ is finitely generated, $\bar\Lambda/\ell\bar\Lambda$ is finite. Therefore $\bar\Lambda$ is finite and (d) holds. $\square$

Remark 4.4.6. Suppose that Λ is finitely generated. The additional assumption that $\ell\Lambda \subseteq \operatorname{supp}_\Lambda(\mathcal{B})$ for some positive integer ℓ holds for many important classes of graded-central-simple Λ-graded algebras such as centreless Lie tori, associative tori, alternative tori and Jordan tori (see Section 9). Thus for any Λ-graded algebra $\mathcal{B}$ in one of these classes, (a) is equivalent to (d) in Proposition 4.4.5.

5. Graded simplicity and centrality for loop algebras

Suppose in this section that Γ is a subgroup of an arbitrary abelian group Λ. Suppose further that $\pi : \Lambda \to \bar\Lambda$ is a (group) epimorphism of Λ onto an abelian group $\bar\Lambda$ such that

$$\ker(\pi) = \Gamma.$$

In this section we investigate centrality and simplicity of the loop algebra $L_\pi(\mathcal{A})$.

5.1. Preliminary lemmas.

Lemma 5.1.1. *Suppose that $\mathcal{A}$ is a $\bar\Lambda$-graded algebra. Then*

$$\mathcal{A} \text{ is graded-simple} \iff L_\pi(\mathcal{A}) \text{ is graded-simple.}$$

Proof. "$\Leftarrow$" If $\mathcal{A}$ is not graded-simple, then $\mathcal{A}$ has a nonzero proper graded ideal $\mathcal{I}$. In that case, $L_\pi(\mathcal{I})$ is a nonzero proper graded ideal of $L_\pi(\mathcal{A})$, and so $L_\pi(\mathcal{A})$ is not graded-simple.

"$\Rightarrow$" Suppose that $\mathcal{A}$ is graded-simple. Note first that the set $\{ u \in \mathcal{A} \mid u\mathcal{A}+\mathcal{A}u = 0 \}$ is a proper graded ideal of $\mathcal{A}$ and hence it is 0. Thus, if $0 \neq u \in \mathcal{A}$ then $u\mathcal{A} + \mathcal{A}u \neq 0$.

Let $\mathcal{I}$ be a nonzero graded ideal of $L_\pi(\mathcal{A})$. For $\lambda \in \Lambda$ we let

$$\mathcal{S}^{\bar{\lambda}} := \{ u \in \mathcal{A}^{\bar{\lambda}} \mid u \otimes z^{\gamma+\lambda} \in \mathcal{I} \text{ for all } \gamma \in \Gamma \},$$

and we set $\mathcal{S} = \sum_{\bar{\lambda}\in\bar{\Lambda}} \mathcal{S}^{\bar{\lambda}}$. Then $\mathcal{S}$ is a graded ideal of $\mathcal{A}$.

Now choose $0 \neq u_1 \otimes z^\mu \in \mathcal{I}$, where $\mu \in \Lambda$ and $u_1 \in \mathcal{A}^{\bar{\mu}}$. Then by the note at the beginning of this proof, we may choose $\lambda \in \Lambda$ and $u_2 \in \mathcal{A}^{\bar{\lambda}}$ such that $u_1u_2 \neq 0$ or $u_2u_1 \neq 0$. We assume that $u_2u_1 \neq 0$ (the other case being similar). For any $\gamma \in \Gamma$, the element $(u_2 \otimes z^{\gamma+\lambda})(u_1 \otimes z^\mu) = u_2u_1 \otimes z^{\gamma+\lambda+\mu}$ is in $\mathcal{I}$. Hence $u_2u_1 \in \mathcal{S}^{\bar{\lambda}+\bar{\mu}}$ and so $\mathcal{S} \neq 0$. Consequently $\mathcal{S} = \mathcal{A}$. Thus for all $\lambda \in \Lambda$ and all $u \in \mathcal{A}^{\bar{\lambda}}$, we have $u \otimes z^\lambda \in \mathcal{I}$. Therefore $\mathcal{I} = L_\pi(\mathcal{A})$. $\square$

Suppose that $\mathcal{A}$ is a graded-simple $\bar{\Lambda}$-graded algebra. By Lemma 4.2.3(ii), $C(\mathcal{A})$ is a $\bar{\Lambda}$-graded algebra and so $L_\pi(C(\mathcal{A}))$ is a Λ-graded algebra. On the other hand, by Lemma 5.1.1, $L_\pi(\mathcal{A})$ is a graded-simple Λ-graded algebra and so $C(L_\pi(\mathcal{A}))$ is a Λ-graded algebra. We let

$$\psi : L_\pi(C(\mathcal{A})) \to C(L_\pi(\mathcal{A}))$$

be the unique k-linear map such that

$$\big(\psi(c \otimes z^\lambda)\big)(u \otimes z^\mu) = c(u) \otimes z^{\lambda+\mu} \tag{13}$$

for $\lambda, \mu \in \Lambda$, $c \in C(\mathcal{A})^{\bar{\lambda}}$, $u \in \mathcal{A}^{\bar{\mu}}$. It is easy to check that ψ is a homomorphism of Λ-graded algebras.

The first part of the following lemma (with weaker assumptions) was proved for classical loop algebras in [2, Proposition 4.11].

Lemma 5.1.2. *Suppose that $\mathcal{A}$ is a graded-simple $\bar{\Lambda}$-graded algebra. Then the map $\psi : L_\pi(C(\mathcal{A})) \to C(L_\pi(\mathcal{A}))$ defined by (13) is an isomorphism of Λ-graded algebras. Moreover,*

$$\Gamma_\Lambda(L_\pi(\mathcal{A})) = \{ \lambda \in \Lambda \mid \bar{\lambda} \in \Gamma_{\bar{\Lambda}}(\mathcal{A}) \}. \tag{14}$$

Proof. $\ker(\psi)$ is a graded ideal of $L_\pi(C(\mathcal{A}))$ and it is clear that $\ker(\psi)^\lambda = 0$ for $\lambda \in \Lambda$. Thus ψ is a monomorphism.

To see that ψ is onto, let $d \in C(L_\pi(\mathcal{A}))^\lambda$ where $\lambda \in \Lambda$. Then for $\mu \in \Lambda$, there exists a unique map

$$c_\mu : \mathcal{A}^{\bar{\mu}} \to \mathcal{A}^{\bar{\mu}+\bar{\lambda}}$$

such that

$$d(u \otimes z^\mu) = c_\mu(u) \otimes z^{\mu+\lambda} \tag{15}$$

for $u \in \mathcal{A}^{\bar{\mu}}$. Then for $\mu_1, \mu_2 \in \Lambda$, $u_1 \in \mathcal{A}^{\bar{\mu}_1}$, $u_2 \in \mathcal{A}^{\bar{\mu}_2}$, we have

$$\begin{aligned} c_{\mu_1+\mu_2}(u_1u_2) \otimes z^{\mu_1+\mu_2+\lambda} &= d(u_1u_2 \otimes z^{\mu_1+\mu_2}) = (u_1 \otimes z^{\mu_1})d(u_2 \otimes z^{\mu_2}) \\ &= (u_1 \otimes z^{\mu_1})(c_{\mu_2}(u_2) \otimes z^{\mu_2+\lambda}) = u_1c_{\mu_2}(u_2) \otimes z^{\mu_1+\mu_2+\lambda}. \end{aligned}$$

Hence

$$c_{\mu_1+\mu_2}(u_1u_2) = u_1c_{\mu_2}(u_2) \quad \text{and similarly} \quad c_{\mu_1+\mu_2}(u_1u_2) = c_{\mu_1}(u_1)u_2 \tag{16}$$

for $\mu_1, \mu_2 \in \Lambda$, $u_1 \in \mathcal{A}^{\bar{\mu}_1}$, $u_2 \in \mathcal{A}^{\bar{\mu}_2}$. Thus if $\mu_1, \mu_2 \in \Lambda$, $\gamma \in \Gamma$, $u_1 \in \mathcal{A}^{\bar{\mu}_1}$, $u_2 \in \mathcal{A}^{\bar{\mu}_2}$, we have

$$c_{\mu_1+\mu_2+\gamma}(u_1u_2) = c_{(\mu_1+\gamma)+\mu_2}(u_1u_2) = u_1c_{\mu_2}(u_2) = c_{\mu_1+\mu_2}(u_1u_2).$$

Consequently if $\mu \in \Lambda$ and $\gamma \in \Gamma$, the maps $c_{\mu+\gamma}$ and c_μ agree on all elements of $(\mathcal{A}\mathcal{A}) \cap \mathcal{A}^{\bar{\mu}}$. Since $(\mathcal{A}\mathcal{A}) \cap \mathcal{A}^{\bar{\mu}} = \mathcal{A}^{\bar{\mu}}$, it follows that $c_{\mu+\gamma} = c_\mu$ for $\mu \in \Lambda$, $\gamma \in \Gamma$. This equation tells us that there is a unique well-defined map $c \in \mathrm{End}_k(\mathcal{A})$ such that $c(u) = c_\mu(u)$ for $u \in \mathcal{A}^{\bar{\mu}}$ and $\mu \in \Lambda$. Clearly $c \in \mathrm{End}_k(\mathcal{A})^{\bar{\lambda}}$ and then by (16), we have $c \in C(\mathcal{A})^{\bar{\lambda}}$. Hence $c \otimes z^\lambda \in L_\pi(C(\mathcal{A}))^\lambda$. Finally, by (15), we have $d(u \otimes z^\mu) = c(u) \otimes z^{\mu+\lambda}$ for $\mu \in \Lambda$ and $u \in \mathcal{A}^{\bar{\mu}}$. Thus $\psi(c \otimes z^\lambda) = d$.

So we have proved the first statement. Hence we have $C(L_\pi(\mathcal{A})) \simeq_\Lambda L_\pi(C(\mathcal{A})) = \sum_{\lambda\in\Lambda} C(\mathcal{A})^{\bar{\lambda}} \otimes z^\lambda$, which implies (14). □

Lemma 5.1.3. *Suppose that $\mathcal{A}$ is a $\bar{\Lambda}$-graded algebra.*

(i) *If $\mathcal{A}$ is graded-simple, then*

$$\mathcal{A} \text{ is graded-central} \iff L_\pi(\mathcal{A}) \text{ is graded-central.}$$

(ii) *If $\mathcal{A}$ is graded-central-simple, then*

$$\mathcal{A} \text{ is central-simple} \iff \Gamma(L_\pi(\mathcal{A})) = \Gamma.$$

(iii) *If $\mathcal{A}$ is central-simple, then*

$$C(L_\pi(\mathcal{A})) = \mathrm{span}_k\{\, l_{1\otimes z^\gamma} \mid \gamma \in \Gamma \,\} \simeq_\Lambda k[\Gamma], \tag{17}$$

where $l_{1\otimes z^\gamma}$ denotes left multiplication by $1 \otimes z^\gamma$. In particular, the centroid of $L_\pi(\mathcal{A})$ is split

Proof. (i): Suppose that $\mathcal{A}$ is graded-simple. Then $L_\pi(C(\mathcal{A}))^0 = C(\mathcal{A})^{\bar{0}} \otimes z^0$. So $L_\pi(C(\mathcal{A}))^0$ and $C(\mathcal{A})^{\bar{0}}$ are isomorphic as unital k-algebras. By Lemma 5.1.2, $C(L_\pi(\mathcal{A}))^0$ and $C(\mathcal{A})^{\bar{0}}$ are thus isomorphic as unital k-algebras and (i) follows from this.

(ii): Suppose that $\mathcal{A}$ is graded-central-simple. Then,

$$\begin{aligned}
\Gamma(L_\pi(\mathcal{A})) = \Gamma &\iff \Gamma_{\bar{\Lambda}}(\mathcal{A}) = \{\,\bar{0}\,\} && \text{(by (14))}\\
&\iff C(\mathcal{A}) = C(\mathcal{A})^{\bar{0}}\\
&\iff C(\mathcal{A}) = k1 && \text{(since } \mathcal{A} \text{ is graded-central)}\\
&\iff \mathcal{A} \text{ is central}\\
&\iff \mathcal{A} \text{ is central-simple,}
\end{aligned}$$

where the last equivalence follows from the second statement in Lemma 4.2.2.

(iii): Observe that

$$L_\pi(C(\mathcal{A})) = \textstyle\sum_{\lambda\in\Lambda} C(\mathcal{A})^{\bar{\lambda}} \otimes z^\lambda = \sum_{\gamma\in\Gamma} k \otimes z^\gamma = \mathrm{span}_k\{\, 1 \otimes z^\gamma \mid \gamma \in \Gamma \,\}.$$

Hence we have (17) by Lemma 5.1.2. □

5.2. **The classes $\mathfrak{A}(\bar{\Lambda})$ and $\mathfrak{B}(\Lambda, \Gamma)$.** For use here and in the rest of the paper we now introduce a class $\mathfrak{A}(\bar{\Lambda})$ of $\bar{\Lambda}$-graded algebras and a class $\mathfrak{B}(\Lambda, \Gamma)$ of Λ-graded algebras. As the notation suggests, the class $\mathfrak{A}(\bar{\Lambda})$ depends on just the group $\bar{\Lambda}$, while the class $\mathfrak{B}(\Lambda, \Gamma)$ depends both on the group Λ and the subgroup Γ.

Definition 5.2.1. (i) We let $\mathfrak{A}(\bar{\Lambda})$ be the class of $\bar{\Lambda}$-graded algebras $\mathcal{A}$ such that $\mathcal{A}$ is central-simple as an algebra.

(ii) We let $\mathfrak{B}(\Lambda, \Gamma)$ be the class of Λ-graded algebras $\mathcal{B}$ such that $\mathcal{B}$ is graded-central-simple, the centroid of $\mathcal{B}$ is split and $\Gamma(\mathcal{B}) = \Gamma$. Equivalently $\mathfrak{B}(\Lambda, \Gamma)$ is the class of Λ-graded algebras $\mathcal{B}$ such that $\mathcal{B}$ is graded-central-simple and $C(\mathcal{B}) \simeq_\Lambda k[\Gamma]$.

Remark 5.2.2. It is clear that the class $\mathfrak{A}(\bar{\Lambda})$ is closed under graded-isomorphism That is, if $\mathcal{A}$ and $\mathcal{A}'$ are $\bar{\Lambda}$-graded algebras such that $\mathcal{A} \in \mathfrak{A}(\bar{\Lambda})$ and $\mathcal{A} \simeq_{\bar{\Lambda}} \mathcal{A}'$, then we also have $\mathcal{A}' \in \mathfrak{A}(\bar{\Lambda})$. Similarly $\mathfrak{B}(\Lambda, \Gamma)$ is closed under graded-isomorphism.

In the next proposition we use the loop construction and the previous lemmas to establish a relationship between the classes $\mathfrak{A}(\bar{\Lambda})$ and $\mathfrak{B}(\Lambda, \Gamma)$ using the loop construction. This relationship will be explored in more detail in §7.

Proposition 5.2.3. *Let $\mathcal{A}$ be a $\bar{\Lambda}$-graded algebra. Then the following statements are equivalent:*

(a) $\mathcal{A} \in \mathfrak{A}(\bar{\Lambda})$.
(b) $L_\pi(\mathcal{A}) \in \mathfrak{B}(\Lambda, \Gamma)$.
(c) $L_\pi(\mathcal{A})$ *is a graded-central-simple with central grading group* Γ.

Proof. "(a) ⇒ (b)" follows from Lemmas 5.1.1 and Lemmas 5.1.3, whereas "(b) ⇒ (c)" is trivial. Finally "(c) ⇒ (a)" follows by Lemmas 5.1.1 and Lemmas 5.1.3. □

6. Central specializations and central images

We have seen in Section 3 how to pass from a $\bar{\Lambda}$-graded algebra to a Λ-graded algebra using the loop construction. In order to provide an inverse for this construction (in a sense to be made precise), we study in this section certain algebra homomorphisms, called central specializations, from Λ-graded algebras onto $\bar{\Lambda}$-graded algebras.

Throughout the section we assume again that Γ is a subgroup of an arbitrary abelian group Λ and that $\pi : \Lambda \to \bar{\Lambda}$ is a group epimorphism such that $\Gamma = \ker(\pi)$.

6.1. **Definitions.**

Definition 6.1.1. Let $\mathcal{B}$ be a Λ-graded algebra and let $\rho \in \mathrm{Alg}(C(\mathcal{B}), k)$. A *$\rho$-specialization* of $\mathcal{B}$ is a nonzero algebra epimorphism $\varphi : \mathcal{B} \to \mathcal{A}$ onto a $\bar{\Lambda}$-graded algebra $\mathcal{A}$ such that the following two conditions hold:

(a) $\varphi(\mathcal{B}^\lambda) \subseteq \mathcal{A}^{\bar{\lambda}}$ for $\lambda \in \Lambda$.
(b) $\varphi(cx) = \rho(c)\varphi(x)$ for $c \in C(\mathcal{B})$, $x \in \mathcal{B}$.

If $\varphi : \mathcal{B} \to \mathcal{A}$ is a ρ-specialization, we call $\mathcal{A}$ a *ρ-image of* $\mathcal{B}$.

A *central specialization* of $\mathcal{B}$ is a map $\varphi : \mathcal{B} \to \mathcal{A}$ that is a ρ-specialization of $\mathcal{B}$ for some $\rho \in \mathrm{Alg}(C(\mathcal{B}), k)$. Similarly a *central image* of $\mathcal{B}$ is a $\bar{\Lambda}$-graded algebra that is a ρ-image of $\mathcal{B}$ for some $\rho \in \mathrm{Alg}(C(\mathcal{B}), k)$.

Of course all of these definitions are made relative to the fixed epimorphism $\pi : \Lambda \to \bar{\Lambda}$.

Remark 6.1.2. Suppose that $\mathcal{A}$ is a finite dimensional simple Lie algebra over an algebraically closed field of characteristic 0. Although not formulated as in Definition 6.1.1, central specializations of classical loop algebras of $\mathcal{A}$ were used by V. Kac in the classification of automorphisms of finite order of $\mathcal{A}$ [14, Theorem 8.6].

Remark 6.1.3. Let $\mathcal{B}$ be a Λ-graded algebra and let $\rho \in \mathrm{Alg}(C(\mathcal{B}), k)$.

(i) Since $\mathcal{B}$ is Λ-graded, $\mathcal{B}$ has a natural $\bar{\Lambda}$-grading defined by

$$\mathcal{B}^{\bar{\lambda}} = \textstyle\sum_{\mu \in \Lambda,\ \bar{\mu} = \bar{\lambda}} \mathcal{B}^\mu \tag{18}$$

for $\lambda \in \Lambda$. (See [18, p. 3].) Then assumption (a) in Definition 6.1.1 says that φ is a $\bar{\Lambda}$-graded map.

(ii) Suppose $\mathcal{B}'$ is another Λ-graded algebra and $\beta : \mathcal{B}' \to \mathcal{B}$ is an isomorphism of Λ-graded algebras. Then β induces an algebra isomorphism

$$C(\beta) : C(\mathcal{B}') \to C(\mathcal{B})$$

defined by $C(\beta)(c') = \beta \circ c' \circ \beta^{-1}$ for $c' \in C(\mathcal{B}')$. It follows that $C(\mathcal{B}') = \bigoplus_{\gamma\in\Gamma} C(\mathcal{B}')^\gamma$ and that $C(\beta)$ is an isomorphism of Λ-graded algebras. Moreover, if $\varphi : \mathcal{B} \to \mathcal{A}$ is a ρ-specialization of $\mathcal{B}$, then $\varphi \circ \beta$ is a $\rho \circ C(\beta)$-specialization of $\mathcal{B}'$. Consequently if $\mathcal{A}$ is a central image of $\mathcal{B}$ then $\mathcal{A}$ is also a central image of $\mathcal{B}'$.

(iii) If $\mathcal{A}$ is a ρ-image of $\mathcal{B}$ and $\mathcal{A}'$ is $\bar\Lambda$-graded algebra such that $\mathcal{A} \simeq_{\bar\Lambda} \mathcal{A}'$, then $\mathcal{A}'$ is also a ρ-image of $\mathcal{A}$.

Example 6.1.4. Let $\mathcal{B}$ be a Λ-graded algebra which satisfies the following conditions:

(a) $C(\mathcal{B})$ is commutative and $C(\mathcal{B}) = \bigoplus_{\gamma\in\Gamma} C(\mathcal{B})^\gamma$ (where $C(\mathcal{B})^\gamma$ is defined by (6) for $\gamma \in \Gamma$).

(b) $\mathcal{B}$ is a nonzero free $C(\mathcal{B})$-module (under the natural action).

(Note that by Lemmas 4.2.3(ii) and 4.4.1, these conditions are satisfied if $\mathcal{B} \in \mathfrak{B}(\Lambda,\Gamma)$.) Suppose $\rho \in \mathrm{Alg}(C(\mathcal{B}), k)$. Let

$$\ker(\rho)\mathcal{B} = \mathrm{span}_k\{cx \mid c \in \ker(\rho),\ x \in \mathcal{B}\}.$$

Then $\ker(\rho)\mathcal{B}$ is an ideal of $\mathcal{B}$ (as an algebra). Also, regarding $\mathcal{B}$ as $\bar\Lambda$-graded as in Remark 6.1.3(i), we have, using assumption (a), that

$$\ker(\rho)\mathcal{B}^{\bar\lambda} \subseteq C(\mathcal{B})\mathcal{B}^{\bar\lambda} \subseteq \big(\textstyle\sum_{\gamma\in\Gamma} C(\mathcal{B})^\gamma\big)\big(\textstyle\sum_{\mu\in\Lambda,\ \bar\mu=\bar\lambda} \mathcal{B}^\mu\big) \subseteq \mathcal{B}^{\bar\lambda}$$

for $\bar\lambda \in \bar\Lambda$. It follows from this that $\ker(\rho)\mathcal{B}$ is a $\bar\Lambda$-graded ideal of $\mathcal{B}$. Thus the quotient algebra

$$\mathcal{B}/\ker(\rho)\mathcal{B}$$

has the natural structure of a $\bar\Lambda$-graded algebra. Observe also that $\ker(\rho) \neq C(\mathcal{B})$ and so, by assumption (b), $\ker(\rho)\mathcal{B} \neq \mathcal{B}$. (Actually it would be enough to assume in place of (b) that $\mathcal{B}$ is a faithfully flat $C(\mathcal{B})$-module [9, §I.3.1].) Thus $\mathcal{B}/\ker(\rho)\mathcal{B} \neq 0$. Finally, let $p_\rho : \mathcal{B} \to \mathcal{B}/\ker(\rho)\mathcal{B}$ be the canonical projection defined by

$$p_\rho(x) = x + \ker(\rho)\mathcal{B}$$

for $x \in \mathcal{B}$. Note that since $(c - \rho(c)1)x \in \ker(\rho)\mathcal{B}$, we have

$$p_\rho(cx) = \rho(c)p_\rho(x)$$

for $c \in C(\mathcal{B})$, $x \in \mathcal{B}$. Thus p_ρ is a ρ-specialization of $\mathcal{B}$. We call p_ρ the *universal ρ-specialization of* $\mathcal{B}$. This terminology is justified by the following lemma.

Lemma 6.1.5. *Let $\mathcal{B}$ be a Λ-graded algebra satisfying assumptions (a) and (b) of Example 6.1.4 and let $\rho \in \mathrm{Alg}(C(\mathcal{B}), k)$. If $\varphi : \mathcal{B} \to \mathcal{A}$ is an arbitrary ρ-specialization of $\mathcal{B}$, then there is a unique $\bar\Lambda$-graded epimorphism $\kappa : \mathcal{B}/\ker(\rho)\mathcal{B} \to \mathcal{A}$ such that $\varphi = \kappa \circ p_\rho$.*

Proof. Now $\ker(\rho)\mathcal{B} \subseteq \ker(\varphi)$. So there is an induced algebra homomorphism $\kappa : \mathcal{B}/\ker(\rho)\mathcal{B} \to \mathcal{A}$ such that

$$\kappa(x + \ker(\rho)\mathcal{B}) = \varphi(x)$$

for $x \in \mathcal{B}$. It is clear that κ is $\bar\Lambda$-graded, and since φ is surjective, κ is surjective. Also, $\varphi = \kappa \circ p_\rho$ by definition of κ. Finally, the uniqueness of κ is clear since p_ρ is surjective. □

Remark 6.1.6. Let $\mathcal{B}$ be a Λ-graded algebra satisfying assumptions (a) and (b) of Example 6.1.4 and let $\rho \in \mathrm{Alg}(C(\mathcal{B}), k)$. Then the universal ρ-specialization of $\mathcal{B}$ has an alternate interpretation that was suggested to us by Ottmar Loos. Indeed, we may regard k as an algebra k_ρ over $C(\mathcal{B})$ by means of the action $(c, a) \mapsto \rho(c)a$ for $c \in C(\mathcal{B})$ and $a \in k$. Then

$$\mathcal{B} \otimes_{C(\mathcal{B})} k_\rho$$

is an algebra over k. Furthermore, as we have seen, it follows from assumption (a) that $\mathcal{B} = \bigoplus_{\bar{\lambda} \in \bar{\Lambda}} \mathcal{B}^{\bar{\lambda}}$ is a decomposition of $\mathcal{B}$ as the direct sum of $C(\mathcal{B})$-modules. Hence $\mathcal{B} \otimes_{C(\mathcal{B})} k_\rho$ is a $\bar{\Lambda}$-graded algebra with $(\mathcal{B} \otimes_{C(\mathcal{B})} k_\rho)^{\bar{\lambda}} := \{ x \otimes 1 \mid x \in \mathcal{B}^{\bar{\lambda}} \}$ for $\bar{\lambda} \in \bar{\Lambda}$. Moreover, $\mathcal{B} \otimes_{C(\mathcal{B})} k_\rho$ is nonzero by assumption (b). Finally one checks easily that the map $x + \ker(\rho)\mathcal{B} \mapsto x \otimes 1$ is a $\bar{\Lambda}$-graded algebra isomorphism of $\mathcal{B}/\ker(\rho)\mathcal{B}$ onto $\mathcal{B} \otimes_{C(\mathcal{B})} k_\rho$. If we regard this map as an identification, then the universal ρ-specialization $p_\rho : \mathcal{B} \to \mathcal{B} \otimes_{C(\mathcal{B})} k_\rho$ is given by $p_\rho(x) = x \otimes 1$ for $x \in \mathcal{B}$.

6.2. **Central specializations and images of algebras in $\mathfrak{B}(\Lambda, \Gamma)$.** If $\mathcal{B}$ is in $\mathfrak{B}(\Lambda, \Gamma)$, then by Lemma 4.3.7 we have $\mathrm{Alg}(C(\mathcal{B}), k) \neq \emptyset$. Moreover, $\mathcal{B}$ satisfies assumptions (a) and (b) of Example 6.1.4, and so, for $\rho \in \mathrm{Alg}(C(\mathcal{B}), k)$, we can construct the universal ρ-specialization $p_\rho : \mathcal{B} \to \mathcal{B}/\ker(\rho)\mathcal{B}$ of $\mathcal{B}$. In part (i) of the next proposition, we see that p_ρ is unique ρ-specialization of $\mathcal{B}$.

Proposition 6.2.1. *Suppose that* $\mathcal{B} \in \mathfrak{B}(\Lambda, \Gamma)$, $\rho \in \mathrm{Alg}(C(\mathcal{B}), k)$, $\mathcal{A}$ *is a* $\bar{\Lambda}$*-graded algebra and* $\varphi : \mathcal{B} \to \mathcal{A}$ *is a* ρ*-specialization of* $\mathcal{B}$*. Then*

(i) *There exists a unique* $\bar{\Lambda}$*-graded isomorphism* $\kappa : \mathcal{B}/\ker(\rho)\mathcal{B} \to \mathcal{A}$ *such that* $\varphi = \kappa \circ p_\rho$.
(ii) *If* X *is a homogeneous* $C(\mathcal{B})$*-basis for* $\mathcal{B}$ *chosen as in Lemma 4.4.1, then* φ *maps* X *bijectively onto a* k*-basis* $\varphi(X)$ *of* $\mathcal{A}$.
(iii) *For* $\lambda \in \Lambda$, φ *restricts to a linear bijection of* $\mathcal{B}^\lambda$ *onto* $\mathcal{A}^{\bar{\lambda}}$.
(iv) $L_\pi(\mathcal{A}) \simeq_\Lambda \mathcal{B}$.
(v) $\mathcal{A} \in \mathfrak{A}(\bar{\Lambda})$.

Proof. We first prove statements (ii)-(v) for the universal ρ-specialization. So in this part of the proof we assume that $\mathcal{A} = \mathcal{B} \otimes_{C(\mathcal{B})} k_\rho$ and $\varphi : \mathcal{B} \to \mathcal{B} \otimes_{C(\mathcal{B})} k_\rho$ is given by $\varphi(x) = x \otimes 1$ (see Remark 6.1.6).

(ii): This is a general property of the tensor product $\mathcal{B} \otimes_{C(\mathcal{B})} k_\rho$ (see [16, Proposition 4.1]).

(iii): This follows from (ii) since we can choose the $C(\mathcal{B})$-basis X for $\mathcal{B}$ so that X contains a k-basis for $\mathcal{B}^\lambda$.

(iv): Define $\omega : \mathcal{B} \to L_\pi(\mathcal{A})$ by

$$\omega(x) = \varphi(x) \otimes z^\lambda$$

for $x \in \mathcal{B}^\lambda$ and $\lambda \in \Lambda$. Then ω is a nonzero homomorphism of Λ-graded algebras. Since $\mathcal{B}$ is graded-simple, ω is a monomorphism. Finally, by (iii), we have $\omega(\mathcal{B}^\lambda) = \mathcal{A}^{\bar{\lambda}} \otimes z^\lambda$ for $\lambda \in \Lambda$, and so ω is surjective.

(v): By (iv), we have $L_\pi(\mathcal{A}) \simeq_\Lambda \mathcal{B}$. So $L_\pi(\mathcal{A})$ is graded-central-simple with central grading group Γ. Thus, by Proposition 5.2.3, $\mathcal{A} \in \mathfrak{A}(\bar{\Lambda})$.

To complete the proof of the proposition, we now assume that $\varphi : \mathcal{B} \to \mathcal{A}$ is an arbitrary ρ-specialization. Since we now know that $\mathcal{B}/\ker(\rho)\mathcal{B}$ is graded-simple (in fact it is central-simple by (v) in the universal case), (i) follows from the last statement of Lemma 6.1.5. (ii)–(v) then follow from the corresponding statements for the universal ρ-specialization. $\square$

By Proposition 6.2.1(i) we have the following:

Corollary 6.2.2. *Suppose* $\mathcal{B} \in \mathfrak{B}(\Lambda, \Gamma)$ *and* $\rho \in \mathrm{Alg}(C(\mathcal{B}), k)$. *Then* $\mathcal{B}/\ker(\rho)\mathcal{B}$ *is the unique* ρ*-image of* $\mathcal{B}$ *up to* $\bar{\Lambda}$*-graded isomorphism.*

6.3. **Similarity for $\bar{\Lambda}$-graded algebras.** In the subsection after this we will look at central images of $\mathcal{B} \in \mathfrak{B}(\Lambda, \Gamma)$ corresponding to different homomorphisms in $\mathrm{Alg}(C(\mathcal{B}), k)$. In preparation for this we look first at a notion of similarity (relative to π) for $\bar{\Lambda}$-graded algebras.

Since $\pi : \Lambda \to \bar{\Lambda}$ is surjective, π has a right inverse as a map of sets. For the rest of §6, *we fix a choice* ξ *of such a right inverse.* So $\xi : \bar{\Lambda} \to \Lambda$ is a map of sets such that

$$\pi \circ \xi = 1_{\bar{\Lambda}}.$$

Definition 6.3.1. Let χ be a character of Γ, i.e. $\chi \in \mathrm{Hom}(\Gamma, k^\times)$. Also let $\mathcal{A}$ be a $\bar{\Lambda}$-graded algebra. We define a $\bar{\Lambda}$-graded algebra $\mathcal{A}_\chi$ as follows. As a $\bar{\Lambda}$-graded vector space $\mathcal{A}_\chi = \mathcal{A}$. Further, the product $\cdot_\chi$ on $\mathcal{A}_\chi$ is defined by

$$u \cdot_\chi v = \chi(\xi(\bar{\lambda}) + \xi(\bar{\mu}) - \xi(\bar{\lambda} + \bar{\mu}))uv$$

for $\bar{\lambda}, \bar{\mu} \in \bar{\Lambda}$, $u \in \mathcal{A}^{\bar{\lambda}}$, $v \in \mathcal{A}^{\bar{\mu}}$. We call $\mathcal{A}_\chi$ the *twist of* $\mathcal{A}$ *by* χ.

Remark 6.3.2. Suppose that χ and $\mathcal{A}$ are as in Definition 6.3.1. It is easy to check that, up to $\bar{\Lambda}$-graded isomorphism, $\mathcal{A}_\chi$ is independent of the choice of the right inverse ξ for π.

Remark 6.3.3. Suppose that $\mathcal{A}$ is a $\bar{\Lambda}$-graded algebra. If $\mathcal{A}$ is a Lie or associative algebra, then one easily checks directly that any twist of $\mathcal{A}$ is a Lie algebra or associative algebra respectively. Note also that if $\mathcal{A}$ is unital then any twist $\mathcal{A}_\chi$ of $\mathcal{A}_\chi$ is unital. Indeed, up to a $\bar{\Lambda}$-isomorphism of $\mathcal{A}_\chi$, we can choose the right inverse ξ with $\xi(\bar{0}) = 0$. In this case, the identity element 1 of $\mathcal{A}$ has $1 \in \mathcal{A}^{\bar{0}}$ and 1 is also an identity of $\mathcal{A}_\chi$.

Twists of $\mathcal{A}$ have the following properties:

Lemma 6.3.4. *Suppose that* $\mathcal{A}$ *is a* $\bar{\Lambda}$*-graded algebra.*

(i) *If* $\mathcal{A}'$ *is a* $\bar{\Lambda}$*-graded algebra such that* $\mathcal{A} \simeq_{\bar{\Lambda}} \mathcal{A}'$, *then* $\mathcal{A}_\chi \simeq_{\bar{\Lambda}} \mathcal{A}'_\chi$ *for* $\chi \in \mathrm{Hom}(\Gamma, k^\times)$.

(ii) *If* $1 \in \mathrm{Hom}(\Gamma, k^\times)$ *is the trivial character (that is* $1(\gamma) = 1$ *for all* $\gamma \in \Gamma$*), then* $\mathcal{A}_1 = \mathcal{A}$.

(iii) *If* $\chi_1, \chi_2 \in \mathrm{Hom}(\Gamma, k^\times)$, *then* $(\mathcal{A}_{\chi_1})_{\chi_2} = \mathcal{A}_{\chi_1\chi_2}$.

(iv) *If* $\chi \in \mathrm{Hom}(\Gamma, k^\times)$ *extends to a character of* Λ, *then* $\mathcal{A}_\chi \simeq_{\bar{\Lambda}} \mathcal{A}$.

(v) *If* k *is algebraically closed, then* $\mathcal{A}_\chi \simeq_{\bar{\Lambda}} \mathcal{A}$ *for any* $\chi \in \mathrm{Hom}(\Gamma, k^\times)$.

Proof. (i) and (ii) are clear, and (iii) is easily checked.

For (iv), suppose that $\chi \in \mathrm{Hom}(\Gamma, k^\times)$ and χ extends to a character ψ of Λ. Then for $\bar{\lambda}, \bar{\mu} \in \bar{\Lambda}$, $u \in \mathcal{A}^{\bar{\lambda}}$, $v \in \mathcal{A}^{\bar{\mu}}$, we have

$$u \cdot_\chi v = \psi(\xi(\bar{\lambda}))\psi(\xi(\bar{\mu}))\psi(\xi(\bar{\lambda} + \mu))^{-1}uv.$$

Hence the map defined by $u \mapsto \psi(\xi(\bar{\lambda}))^{-1}u$ for $\bar{\lambda} \in \bar{\Lambda}$, $u \in \mathcal{A}^{\bar{\lambda}}$ is a $\bar{\Lambda}$-graded isomorphism of $\mathcal{A}$ onto $\mathcal{A}_\chi$.

For (v), suppose that k is algebraically closed. Then by [22, Lemma 1.2.7], any character of Γ extends to a character of Λ. So (v) follows from (iv). $\square$

Remark 6.3.5. Twists of $\mathcal{A}$ and the preceding lemma have a cohomological interpretation using the exact sequence $H^1(\Lambda, k^\times) \to H^1(\Gamma, k^\times) \to H^2(\bar{\Lambda}, k^\times)$ arising from the exact sequence $0 \to \Gamma \to \Lambda \to \bar{\Lambda} \to 0$ (with trivial actions on $k^\times$) [23, §2.6]. Since we will not make use of this here, we omit the details.

Definition 6.3.6. If $\mathcal{A}$ and $\mathcal{A}'$ are $\bar{\Lambda}$-graded algebras, we say that $\mathcal{A}$ and $\mathcal{A}'$ are *similar relative to* π, written $\mathcal{A} \sim_\pi \mathcal{A}'$, if $\mathcal{A}' \simeq_{\bar{\Lambda}} \mathcal{A}_\chi$ for some $\chi \in \mathrm{Hom}(\Gamma, k^\times)$.

Remark 6.3.7. (i) The relation $\sim_\pi$ depends on the group epimorphism $\pi : \Lambda \to \bar{\Lambda}$ (with kernel Γ) but not on the choice of the right inverse ξ for π. (See Remark 6.3.2.)

(ii) It follows from parts (i)–(iii) of Lemma 6.3.4 that $\sim_\pi$ is an equivalence relation on the class of $\bar{\Lambda}$-graded algebras.

(iii) Suppose that $\mathcal{A}$ and $\mathcal{A}'$ are $\bar{\Lambda}$-graded algebras. By Lemma 6.3.4(ii) we see that

$$\mathcal{A} \simeq_{\bar{\Lambda}} \mathcal{A}' \Rightarrow \mathcal{A} \sim_\pi \mathcal{A}'.$$

Moreover, if k is algebraically closed, then by Lemma 6.3.4 (v), we have

$$\mathcal{A} \simeq_{\bar{\Lambda}} \mathcal{A}' \iff \mathcal{A} \sim_\pi \mathcal{A}'.$$

6.4. **Comparing central images.** We now look at central images of $\mathcal{B} \in \mathfrak{B}(\Lambda, \Gamma)$ corresponding to different homomorphisms in $\mathrm{Alg}(C(\mathcal{B}), k)$.

Lemma 6.4.1. *Suppose that* $\mathcal{B} \in \mathfrak{B}(\Lambda, \Gamma)$ *and* $\rho \in \mathrm{Alg}(C(\mathcal{B}), k)$.

(i) *For* $\chi \in \mathrm{Hom}(\Gamma, k^\times)$, *set* $\rho_\chi(c) = \chi(\gamma)\rho(c)$ *for* $\gamma \in \Gamma$ *and* $c \in C(\mathcal{B})^\gamma$. *Then* $\chi \to \rho_\chi$ *is a bijection of* $\mathrm{Hom}(\Gamma, k^\times)$ *onto* $\mathrm{Alg}(C(\mathcal{B}), k)$.
(ii) *If* $\chi \in \mathrm{Hom}(\Gamma, k^\times)$ *and* $\mathcal{A}$ *is a* ρ*-image of* $\mathcal{B}$, *then* $\mathcal{A}_\chi$ *is a* ρ_χ*-image of* $\mathcal{B}$.
(iii) *If* $\chi \in \mathrm{Hom}(\Gamma, k^\times)$, $\mathcal{A}$ *is a* ρ*-image of* $\mathcal{B}$ *and* $\mathcal{A}'$ *is a* ρ_χ*-image of* $\mathcal{B}$, *then* $\mathcal{A}' \simeq_{\bar{\Lambda}} \mathcal{A}_\chi$.

Proof. (i) is clear since $C(\mathcal{B})$ is isomorphic to $k[\Gamma]$ by Lemma 4.3.7.

(ii): Suppose $\chi \in \mathrm{Hom}(\Gamma, k^\times)$ and $\mathcal{A}$ is a ρ-image of $\mathcal{B}$. Then we have a ρ-specialization $\varphi : \mathcal{B} \to \mathcal{A}$. Define $\varphi' : \mathcal{B} \to \mathcal{A}_\chi$ by

$$\varphi'(x) = \chi(\lambda - \xi(\bar{\lambda}))\varphi(x)$$

for $\lambda \in \Lambda$, $x \in \mathcal{B}^\lambda$. Then φ' is nonzero, surjective and $\bar{\Lambda}$-graded. Also if $\lambda, \mu \in \Lambda$, $x \in \mathcal{B}^\lambda$ and $y \in \mathcal{B}^\mu$, we have

$$\begin{aligned}
\varphi'(xy) &= \chi(\lambda + \mu - \xi(\bar{\lambda} + \bar{\mu}))\ \varphi(xy) \\
&= \chi(\lambda + \mu - \xi(\bar{\lambda} + \bar{\mu}))\ \varphi(x)\varphi(y) \\
&= \frac{\chi(\lambda + \mu - \xi(\bar{\lambda} + \bar{\mu}))}{\chi(\xi(\bar{\lambda}) + \xi(\bar{\mu}) - \xi(\bar{\lambda} + \bar{\mu}))}\ \varphi(x) \cdot_\chi \varphi(y) \\
&= \chi(\lambda + \mu - \xi(\bar{\lambda}) - \xi(\bar{\mu}))\ \varphi(x) \cdot_\chi \varphi(y) \\
&= \chi(\lambda - \xi(\bar{\lambda}))\chi(\mu - \xi(\bar{\mu}))\ \varphi(x) \cdot_\chi \varphi(y) \\
&= \varphi'(x) \cdot_\chi \varphi'(y).
\end{aligned}$$

Further if $\gamma \in \Gamma$, $\lambda \in \Lambda$, $c \in C(\mathcal{B})^\gamma$ and $x \in \mathcal{B}^\lambda$, we have

$$\begin{aligned}\varphi'(cx) &= \chi(\lambda + \gamma - \xi(\bar{\lambda} + \bar{\gamma}))\ \varphi(cx) \\ &= \chi(\lambda + \gamma - \xi(\bar{\lambda}))\rho(c)\ \varphi(x) \\ &= \chi(\gamma)\chi(\lambda - \xi(\bar{\lambda}))\rho(c)\ \varphi(x) \\ &= \rho_\chi(c)\varphi'(x).\end{aligned}$$

Thus φ' is a ρ_χ-specialization and so $\mathcal{A}_\chi$ is a ρ_χ-image of $\mathcal{A}$.

(iii): This follows from (ii) and the uniqueness of the ρ_χ-image (established in Corollary 6.2.2). □

Proposition 6.4.2. *Suppose that $\mathcal{B} \in \mathfrak{B}(\Lambda, \Gamma)$ and $\mathcal{A}$ is a central image of $\mathcal{B}$. If $\mathcal{A}'$ is a $\bar{\Lambda}$-graded algebra, then $\mathcal{A}'$ is a central image of $\mathcal{B}$ if and only if $\mathcal{A}' \sim_\pi \mathcal{A}$.*

Proof. We are given that $\mathcal{A}$ is a ρ-image of $\mathcal{B}$ for some $\rho \in \mathrm{Alg}(C(\mathcal{B}), k)$.

"$\Rightarrow$" If $\mathcal{A}'$ is a ρ'-image of $\mathcal{B}$ for some $\rho' \in \mathrm{Alg}(C(\mathcal{B}), k)$, then $\mathcal{A}' \simeq_{\bar{\Lambda}} \mathcal{A}_\chi$ for some $\chi \in \mathrm{Hom}(\Gamma, k^\times)$ by parts (i) and (iii) of Lemma 6.4.1. So $\mathcal{A} \sim_\pi \mathcal{A}'$.

"$\Leftarrow$" Suppose that $\mathcal{A}' \simeq \mathcal{A}_\chi$ for some $\chi \in \mathrm{Hom}(\Gamma, k^\times)$.By Lemma 6.4.1(ii), $\mathcal{A}_\chi$, and hence $\mathcal{A}'$, is a ρ_χ-image of $\mathcal{B}$. □

6.5. **A characterization of central images.** The next example shows that $\mathcal{A}$ is a central image of $L_\pi(\mathcal{A})$ for $\mathcal{A} \in \mathfrak{A}(\bar{\Lambda})$.

Example 6.5.1. Suppose that $\mathcal{A} \in \mathfrak{A}(\bar{\Lambda})$ and let $\mathcal{B} = L_\pi(\mathcal{A})$. Then by (17), we have $C(\mathcal{B}) = \mathrm{span}_k\{\, l_{1\otimes z^\gamma} \mid \gamma \in \Gamma \,\} \simeq_\Lambda k[\Gamma]$, and so we may define $\rho \in \mathrm{Alg}(C(\mathcal{B}), k)$ by

$$\rho(l_{1\otimes z^\gamma}) = 1$$

for $\gamma \in \Gamma$. We call ρ the *augmentation homomorphism*. Define $\varphi : \mathcal{B} \to \mathcal{A}$ by

$$\varphi(u \otimes z^\lambda) = u$$

for $u \in \mathcal{A}^\lambda$ and $\lambda \in \Lambda$. Then one checks easily that φ is a ρ-specialization. Therefore, $\mathcal{A}$ is a ρ-image of $L_\pi(\mathcal{A})$.

We can now characterize the central images of a given $\mathcal{B} \in \mathfrak{B}(\Lambda, \Gamma)$ using loop algebras.

Proposition 6.5.2. *Suppose that $\mathcal{B} \in \mathfrak{B}(\Lambda, \Gamma)$ and $\mathcal{A}$ is a $\bar{\Lambda}$-graded algebra. Then the following statements are equivalent:*

(a) $L_\pi(\mathcal{A}) \simeq_\Lambda \mathcal{B}$.
(b) *$\mathcal{A}$ is a central image of $\mathcal{B}$.*
(c) $\mathcal{A} \simeq_{\bar{\Lambda}} \mathcal{B}/\ker(\rho)\mathcal{B}$ *for some* $\rho \in \mathrm{Alg}(C(\mathcal{B}), k)$.

Moreover if (a), (b) or (c) hold, then $\mathcal{A} \in \mathfrak{A}(\bar{\Lambda})$.

Proof. If (a) holds then $\mathcal{A} \in \mathfrak{A}(\bar{\Lambda})$ by Proposition 5.2.3. Thus is suffices to show that (a), (b) and (c) are equivalent. But (b) and (c) are equivalent by the uniqueness of the ρ-image of $\mathcal{B}$ for $\rho \in \mathrm{Alg}(C(\mathcal{B}), k)$ (see Corollary 6.2.2). Moreover "(b) $\Rightarrow$ (a)" follows from Proposition 6.2.1(iv). Finally, "(a) $\Rightarrow$ (b)" follows from Example 6.5.1. □

Corollary 6.5.3. *Suppose that $\mathcal{A} \in \mathfrak{A}(\bar{\Lambda})$ and $\mathcal{A}'$ is a $\bar{\Lambda}$-graded algebra. If $\mathcal{A} \sim_\pi \mathcal{A}'$ then $\mathcal{A}' \in \mathfrak{A}(\bar{\Lambda})$.*

Proof. By Proposition 5.2.3, $L_\pi(\mathcal{A}) \in \mathfrak{B}(\Lambda, \Gamma)$. Then, by Proposition 6.5.2, $\mathcal{A}$ is a central image of $L_\pi(\mathcal{A})$. Thus, since $\mathcal{A} \sim_\pi \mathcal{A}'$, we know by Proposition 6.4.2 that $\mathcal{A}'$ is a central image of $L_\pi(\mathcal{A})$. So, by Proposition 6.5.2, $\mathcal{A}' \in \mathfrak{A}(\bar{\Lambda})$. □

7. The correspondence

Suppose again in this section that Γ is a subgroup of an arbitrary abelian group Λ, and that $\pi : \Lambda \to \bar{\Lambda}$ is a group epimorphism with kernel Γ.

7.1. **The correspondence theorem.**

We can now combine the results from the previous sections to prove our main theorem about the loop algebra construction. This theorem tells us that the loop construction induces a correspondence between similarity classes of $\bar{\Lambda}$-graded algebras in $\mathfrak{A}(\bar{\Lambda})$ and graded-isomorphism classes of Λ-graded algebras in $\mathfrak{B}(\Lambda, \Gamma)$. The inverse correspondence is induced by central specialization.

Theorem 7.1.1 (Correspondence Theorem)**.** *Let Γ be a subgroup of Λ and let $\pi : \Lambda \to \bar{\Lambda}$ be a group epimorphism such that $\ker(\pi) = \Gamma$. Let $\mathfrak{A}(\bar{\Lambda})$ be the class of $\bar{\Lambda}$-graded algebras $\mathcal{A}$ such that $\mathcal{A}$ is central-simple as an algebra. Let $\mathfrak{B}(\Lambda, \Gamma)$ be the class of Λ-graded algebras $\mathcal{B}$ such that $\mathcal{B}$ is graded-central-simple, $C(\mathcal{B})$ is split and $\Gamma(\mathcal{B}) = \Gamma$. For a $\bar{\Lambda}$-graded algebra $\mathcal{A}$, let $L_\pi(\mathcal{A}) = \sum_{\lambda \in \Lambda} \mathcal{A}^{\bar{\lambda}} \otimes z^\lambda$, where $\bar{\lambda} = \pi(\lambda)$ (see Definition 3.1.1).*

- (i) *If $\mathcal{A} \in \mathfrak{A}(\bar{\Lambda})$, then $L_\pi(\mathcal{A}) \in \mathfrak{B}(\Lambda, \Gamma)$.*
- (ii) *If $\mathcal{B} \in \mathfrak{B}(\Lambda, \Gamma)$, then there exists $\mathcal{A} \in \mathfrak{A}(\bar{\Lambda})$ such that $L_\pi(\mathcal{A}) \simeq_\Lambda \mathcal{B}$. Moreover the $\bar{\Lambda}$-graded algebras $\mathcal{A}$ with this property are precisely the central images of $\mathcal{B}$.*
- (iii) *If $\mathcal{A}, \mathcal{A}' \in \mathfrak{A}(\bar{\Lambda})$, then $L_\pi(\mathcal{A}) \simeq_\Lambda L_\pi(\mathcal{A}')$ if and only if $\mathcal{A} \sim_\pi \mathcal{A}'$.*
- (iv) *If $\mathcal{A} \in \mathfrak{A}(\bar{\Lambda})$, $\mathcal{B} \in \mathfrak{B}(\Lambda, \Gamma)$ and $\mathcal{B} \simeq_\Lambda L_\pi(\mathcal{A})$, then $\mathcal{A}$ is finite dimensional if and only if $\mathcal{B}$ is fgc (finitely generated as a module over its centroid).*

Proof. Now (i) follows from Proposition 5.2.3, and (ii) follows from Proposition 6.5.2. To prove (iii), suppose that $\mathcal{A}, \mathcal{A}' \in \mathfrak{A}(\bar{\Lambda})$. Then $\mathcal{A}$ is a central image of $L_\pi(\mathcal{A})$ by Proposition 6.5.2. So

$$\begin{aligned} \mathcal{A} \sim_\pi \mathcal{A}' &\iff \mathcal{A}' \text{ is a central image of } L_\pi(\mathcal{A}) && \text{(by Proposition 6.4.2)} \\ &\iff L_\pi(\mathcal{A}) \simeq_\Lambda L_\pi(\mathcal{A}') && \text{(by Proposition 6.5.2).} \end{aligned}$$

Finally, (iv) follows from (ii) and Proposition 6.2.1(ii). □

Definition 7.1.2. Let Γ, Λ and $\pi : \Lambda \to \bar{\Lambda}$ be as in the Correspondence Theorem. If $\mathcal{A} \in \mathfrak{A}(\bar{\Lambda})$ and $\mathcal{B} \in \mathfrak{B}(\Lambda, \Gamma)$, we say that $\mathcal{A}$ and $\mathcal{B}$ *correspond* (under π) if $L_\pi(\mathcal{A}) \simeq_\Lambda \mathcal{B}$. By the theorem, $\mathcal{A}$ and $\mathcal{B}$ correspond under π if and only if $\mathcal{A}$ is a central image of $\mathcal{B}$ (relative to π).

Remark 7.1.3. Theorem 7.1.1 has a categorical formulation. To describe this we define two categories $\mathfrak{A}_{\text{cat}}$ and $\mathfrak{B}_{\text{cat}}$ and a functor $F : \mathfrak{A}_{\text{cat}} \to \mathfrak{B}_{\text{cat}}$. Although we suppress this from the notation, $\mathfrak{A}_{\text{cat}}$, $\mathfrak{B}_{\text{cat}}$ and F depend on the group epimorphism $\pi : \Lambda \to \bar{\Lambda}$ with kernel Γ. (The category $\mathfrak{A}_{\text{cat}}$ and the functor F, but not the category $\mathfrak{B}_{\text{cat}}$, also depend on the choice of a fixed right inverse ξ for π as in §6.)

The objects in $\mathfrak{A}_{\text{cat}}$ are the $\bar{\Lambda}$-graded algebras in $\mathfrak{A}(\bar{\Lambda})$. A morphism in $\mathfrak{A}_{\text{cat}}$ from $\mathcal{A}$ to $\mathcal{A}'$ is a pair (η, χ), where η is a $\bar{\Lambda}$-graded-isomorphism from $\mathcal{A}$ to $\mathcal{A}'_\chi$ and $\chi \in \text{Hom}(\Gamma, k^\times)$.

The objects in $\mathfrak{B}_{\mathrm{cat}}$ are the pairs $(\mathcal{B}, \rho)$, where $\mathcal{B}$ is a Λ-graded algebra in $\mathfrak{B}(\Lambda, \Gamma)$ and $\rho \in \mathrm{Alg}(C(\mathcal{B}), k)$. A morphism in $\mathfrak{B}_{\mathrm{cat}}$ from $(\mathcal{B}, \rho)$ to $(\mathcal{B}', \rho')$ is a Λ-graded isomorphism $\psi : \mathcal{B} \to \mathcal{B}'$.

If $\mathcal{A}$ is an object in $\mathfrak{A}_{\mathrm{cat}}$, we let $F(\mathcal{A}) = (L_\pi(\mathcal{A}), \rho)$, where $\rho : C(L_\pi(\mathcal{A})) \to k$ is the augmentation homomorphism (see Example 6.5.1). If (η, χ) is a morphism from $\mathcal{A}$ to $\mathcal{A}'$ in $\mathfrak{A}_{\mathrm{cat}}$, we let $F(\eta, \chi)$ be the map from $L_\pi(\mathcal{A})$ to $L_\pi(\mathcal{A}')$ given by $x \otimes z^\lambda \mapsto \frac{1}{\chi(\lambda - \xi(\bar{\lambda}))} \eta(x) \otimes z^\lambda$ for $x \in \mathcal{A}^\lambda$.

Then using the results (and their proofs) from Sections 6.2, 6.4 and 6.5 one can show that F is an equivalence of categories. We leave the details of this to the reader.

Remark 7.1.4. Suppose that k is algebraically closed of characteristic 0 and Λ is finite, and assume that the graded algebras $\mathcal{A}$ and $\mathcal{B}$ are finite dimensional. Then, (i) and the first statement of (ii) in Theorem 7.1.1 were proved by Bahturin, Sehgal and Zaicev in [3, Theorem 7] (although the description of the loop algebra used there is not the same as the one used here).

7.2. **A quantum torus example.**

Example 7.2.1. Suppose that m is a positive integer and that k contains a primitive m^{th} root of unity ζ_m. Let $\Lambda = \mathbb{Z}^2$ and $\bar{\Lambda} = \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$ and let $\pi : \Lambda \to \bar{\Lambda}$ be the natural map.

Let $\mathbf{q} = \begin{bmatrix} 1 & \zeta_m \\ \zeta_m^{-1} & 1 \end{bmatrix}$ and let $\mathcal{B} = k_{\mathbf{q}}$ be the *quantum torus* determined by $\mathbf{q}$. Thus, by definition, $\mathcal{B}$ is the unital associative algebra determined by the generators $x_1, x_1^{-1}, x_2, x_2^{-1}$ subject to the relations

$$x_i x_i^{-1} = x_i^{-1} x_i = 1 \text{ for } i = 1, 2 \quad \text{and} \quad x_2 x_1 = \zeta_m x_1 x_2.$$

Then $\mathcal{B} = \bigoplus_{\lambda \in \Lambda} \mathcal{B}^\lambda$ is a Λ-graded algebra with $\mathcal{B}^{(\ell_1, \ell_2)} = k x_1^{\ell_1} x_2^{\ell_2}$ for $(\ell_1, \ell_2) \in \Lambda$.

We identify the centroid $C(\mathcal{B})$ of $\mathcal{B}$ with the centre of $\mathcal{B}$ (see Remark 4.1.2) in which case we have

$$C(\mathcal{B}) = k[c_1^{\pm 1}, c_2^{\pm 1}],$$

where $c_1 = x_1^m$ and $c_2 = x_2^m$. Thus, $\mathcal{B}$ is a graded-central-simple Λ-graded algebra, $C(\mathcal{B})$ is split and $\Gamma(\mathcal{B}) = m\mathbb{Z} \oplus m\mathbb{Z}$. In other words, $\mathcal{B} \in \mathfrak{B}(\Lambda, m\mathbb{Z} \oplus m\mathbb{Z})$.

Next the elements of $\mathrm{Alg}(C(\mathcal{B}), k)$ are the maps of the form

$$\rho_{a_1, a_2} : f(c_1, c_2) \mapsto f(a_1, a_2)$$

for $f(c_1, c_2) \in C(\mathcal{B})$, where $a_1, a_2 \in k^\times$. Let

$$\mathcal{A}_{a_1, a_2} = \mathcal{B} / \ker(\rho_{a_1, a_2}) \mathcal{B}$$

(as in Example 6.1.4) for $a_1, a_2 \in k^\times$. Then, $\mathcal{A}_{a_1, a_2}$ can be identified with the unital associative algebra determined by the generators y_1, y_2 subject to the relations

$$y_1^m = a_1, \quad y_2^m = a_2 \quad \text{and} \quad y_2 y_1 = \zeta_m y_1 y_2.$$

That is, $\mathcal{A}_{a_1, a_2}$ is the *power norm residue algebra* over k determined by a_1, a_2, m and ζ_m (see [10, §11]). Moreover, the $\bar{\Lambda}$-grading on $\mathcal{A}_{a_1, a_2}$ is determined by the conditions

$$\deg(y_1) = (\bar{1}, \bar{0}) \quad \text{and} \quad \deg(y_2) = (\bar{0}, \bar{1}).$$

Thus, by Proposition 6.5.2, the central images of $\mathcal{B}$ are precisely the power norm residue algebras $\mathcal{A}_{a_1, a_2}$, $a_1, a_2 \in k^\times$ (with the indicated $\bar{\Lambda}$-gradings). Consequently,

these algebras are precisely the $\bar{\Lambda}$-graded algebras in $\mathfrak{A}(\bar{\Lambda})$ that correspond to $\mathcal{B}$ under π.

It follows from Theorem 7.1.1(ii), that we have

$$L_\pi(\mathcal{A}_{a_1,a_2}) \simeq_\Lambda \mathcal{B}$$

for $a_1, a_2 \in k^\times$. Consequently (see (4)), we have

$$M_{(m,m)}(\mathcal{A}_{a_1,a_2}, \sigma_1, \sigma_2) \simeq_\Lambda \mathcal{B}, \tag{19}$$

where σ_1, σ_2 are the automorphisms of $\mathcal{A}_{a_1,a_2}$ determined by the conditions:

$$\sigma_1(y_1) = \zeta_m y_1, \quad \sigma_1(y_2) = y_2, \quad \sigma_2(y_1) = y_1 \quad \text{and} \quad \sigma_2(y_2) = \zeta_m y_2.$$

Remark 7.2.2. The fact that the quantum torus $k_{\mathbf{q}}$ in Example 7.2.1 is a multiloop algebra was previously observed in [2, Example 9.8], where it was shown directly that $\mathcal{B}$ is a multiloop algebra based on $\mathcal{A}_{1,1}$ (the algebra of $m \times m$-matrices over k). The isomorphism (19) shows more generally that $\mathcal{B}$ is a multiloop algebra based on any power norm residue algebra $\mathcal{A}_{a_1,a_2}$ and it places this fact in the much more general context of the correspondence theorem.

8. Multiloop realization of graded algebras

In this section we prove our main results about multiloop realizations of graded-central-simple algebras (see Theorem 8.3.2 and Corollary 8.3.5). Throughout the section we assume that n is an integer ≥ 1, Λ is a free abelian group of rank n and k is an algebraically closed field of characteristic zero.

In view of our assumptions on k, k contains a primitive root ℓ^{th} of unity ζ_ℓ for all positive integers ℓ. We assume that we have made a fixed compatible choice of these roots of unity in the sense that

$$\zeta_{m\ell}^m = \zeta_\ell \tag{20}$$

for all $\ell, m \geq 1$. (This is always possible.) We use these roots of unity in the construction of multiloop algebras.

8.1. $(\mathbf{m}', \mathbf{m})$-admissible matrices.

We begin by describing some terminology that will be useful in the study of multiloop algebras.

Definition 8.1.1. Let $\mathbf{m} = (m_1, \dots, m_n)$ and $\mathbf{m}' = (m'_1, \dots, m'_n)$ be n-tuples of positive integers. Let $D_{\mathbf{m}} = \operatorname{diag}(m_1, \dots, m_n)$ and $D_{\mathbf{m}'} = \operatorname{diag}(m'_1, \dots, m'_n)$.

(i) If P is an $n \times n$-matrix with rational entries, we define the $(\mathbf{m}', \mathbf{m})$-*transpose* of P to be the $n \times n$-matrix

$$D_{\mathbf{m}'} P^t D_{\mathbf{m}}^{-1} = \left(\frac{m'_i}{m_j} p_{ji}\right),$$

where P^t denotes the (usual) transpose of P.

(ii) Recall that $\mathrm{GL}_n(\mathbb{Z})$ is the group of all $n \times n$ matrices with integer entries and determinant ± 1. If $P \in \mathrm{GL}_n(\mathbb{Z})$, we say that P is $(\mathbf{m}', \mathbf{m})$-*admissible* if the $(\mathbf{m}', \mathbf{m})$-transpose of P is in $\mathrm{GL}_n(\mathbb{Z})$. Note that if P is $(\mathbf{m}', \mathbf{m})$-admissible with $(\mathbf{m}', \mathbf{m})$-transpose Q, then Q is $(\mathbf{m}, \mathbf{m}')$-admissible with $(\mathbf{m}, \mathbf{m}')$-transpose P and P^{-1} is $(\mathbf{m}, \mathbf{m}')$-admissible with $(\mathbf{m}, \mathbf{m}')$-transpose Q^{-1}.

The following lemma gives a useful characterization of $(\mathbf{m}', \mathbf{m})$-admissible matrices. In this lemma (and later) we identify $P \in \mathrm{GL}_n(\mathbb{Z})$ with the automorphism of $\mathbb{Z}^n$ given by

$$\boldsymbol{\ell} \xrightarrow{P} \boldsymbol{\ell} P^t. \tag{21}$$

(We use the right transpose action since we are viewing elements of $\mathbb{Z}^n$ as row vectors.)

Lemma 8.1.2. *Let $\mathbf{m} = (m_1, \ldots, m_n)$ and $\mathbf{m}' = (m'_1, \ldots, m'_n)$ be n-tuples of positive integers and let $P \in GL_n(\mathbb{Z})$. Set $\bar{\Lambda} = \mathbb{Z}/(m_1) \oplus \ldots \oplus \mathbb{Z}/(m_n)$ and $\bar{\Lambda}' = \mathbb{Z}/(m'_1) \oplus \ldots \oplus \mathbb{Z}/(m'_n)$. Then P induces an isomorphism of $\bar{\Lambda}'$ onto $\bar{\Lambda}$ if and only if P is $(\mathbf{m}', \mathbf{m})$-admissible.*

Proof. We note that the kernel of natural homomorphism $\mathbb{Z}^n \to \bar{\Lambda}$ is $\mathbb{Z}^n D_{\mathbf{m}}$. Thus, P induces an isomorphism of $\bar{\Lambda}'$ onto $\bar{\Lambda}$ if and only if $\mathbb{Z}^n D_{\mathbf{m}'} P^t = \mathbb{Z}^n D_{\mathbf{m}}$, or equivalently $\mathbb{Z}^n D_{\mathbf{m}'} P^t D_{\mathbf{m}}^{-1} = \mathbb{Z}^n$. But the last condition is equivalent to $D_{\mathbf{m}'} P^t D_{\mathbf{m}}^{-1} \in GL_n(\mathbb{Z})$. □

8.2. **Properties of multiloop algebras.** In this subsection, we prove two basic propositions about multiloop algebras.

Proposition 8.2.1. *Let $\mathbf{m} = (m_1, \ldots, m_n)$ and $\mathbf{m}' = (m'_1, \ldots, m'_n)$ be n-tuples of positive integers and suppose that $P = (p_{ij}) \in GL_n(\mathbb{Z})$ is $(\mathbf{m}', \mathbf{m})$-admissible with*

$$Q := D_{\mathbf{m}'} P^t D_{\mathbf{m}}^{-1} \in GL_n(\mathbb{Z}). \tag{22}$$

Let $\sigma_1, \ldots, \sigma_n$ be commuting automorphisms of an algebra $\mathcal{A}$ such that $\sigma_i^{m_i} = 1$ for $1 \le i \le n$, and let

$$\sigma'_i = \textstyle\prod_{j=1}^n \sigma_j^{p_{ji}}$$

for $1 \le i \le n$. Then $\sigma'_1, \ldots, \sigma'_n$ are commuting automorphisms of $\mathcal{A}$ such that $\sigma_i'^{m'_i} = 1$ for $1 \le i \le n$. Moreover, we have

$$M_{\mathbf{m}'}(\mathcal{A}, \sigma'_1, \ldots, \sigma'_n) \simeq_{\mathbb{Z}^n} M_{\mathbf{m}}(\mathcal{A}, \sigma_1, \ldots, \sigma_n)_R \tag{23}$$

where $R = Q^{-1} \in GL_n(\mathbb{Z})$, and hence

$$M_{\mathbf{m}'}(\mathcal{A}, \sigma'_1, \ldots, \sigma'_n) \simeq_{ig} M_{\mathbf{m}}(\mathcal{A}, \sigma_1, \ldots, \sigma_n). \tag{24}$$

(Here we are using the notation in Definitions 2.1.3(ii) and 2.1.4, as well as the identification $\mathrm{Aut}(\mathbb{Z}^n) = \mathrm{GL}_n(\mathbb{Z})$ in (21).)

Proof. As in Remark 3.2.3(iii), set $G = \langle \sigma_1, \ldots, \sigma_n \rangle$ and $\sigma^{\boldsymbol{\ell}} = \prod_{i=1}^n \sigma_i^{l_i} \in G$ for $\boldsymbol{\ell} = (l_1, \ldots l_n) \in \mathbb{Z}^n$. Similarly, set

$$\sigma'^{\boldsymbol{\ell}} = \textstyle\prod_{i=1}^n \sigma_i'^{l_i} = \prod_{i=1}^n \prod_{j=1}^n \sigma_j^{l_i p_{ji}} = \prod_{j=1}^n \sigma_j^{\sum_{i=1}^n l_i p_{ji}} = \sigma^{\boldsymbol{\ell} P^t}. \tag{25}$$

Since P is invertible it follows that we also have $G = \langle \sigma'_1, \ldots, \sigma'_n \rangle$.

Now our assumption that $\sigma_i^{m_i} = 1$ for $1 \le i \le n$ is equivalent to $\sigma^{\mathbb{Z}^n D_{\mathbf{m}}} = 1$. But we have

$$\sigma'^{\mathbb{Z}^n D_{\mathbf{m}'}} = \sigma^{\mathbb{Z}^n D_{\mathbf{m}'} P^t} = \sigma^{\mathbb{Z}^n Q D_{\mathbf{m}}} = 1,$$

and so $\sigma_i'^{m'_i} = 1$ for $1 \le i \le n$.

It remains to prove (23) (since (24) follows from (23)). For this purpose, we let $\mathcal{B} = M_{\mathbf{m}}(\mathcal{A}, \sigma_1, \ldots, \sigma_n)$ and $\mathcal{B}' = M_{\mathbf{m}'}(\mathcal{A}, \sigma'_1, \ldots, \sigma'_n)$, in which case we must prove that $\mathcal{B}_R \simeq_{\mathbb{Z}^n} \mathcal{B}'$ or equivalently

$$\mathcal{B} \simeq_{\mathbb{Z}^n} \mathcal{B}'_Q \tag{26}$$

(since $Q = R^{-1}$).

Let $S = k[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$. Then $\mathcal{A}\otimes_k S$ is a $\mathbb{Z}^n$-graded algebra (with the grading determined by the natural grading on S), and both $\mathcal{B}$ and $\mathcal{B}'$ are graded subalgebras of $\mathcal{A}\otimes_k S$. Next we define $\gamma_Q \in \mathrm{Aut}_k(S)$ by $\gamma_Q(z^{\boldsymbol{\ell}}) = z^{\boldsymbol{\ell} Q^t}$ where $z^{\boldsymbol{\ell}} = z_1^{l_1} \dots z_n^{l_n}$ for $\boldsymbol{\ell} = (l_1, \dots l_n)$. Then $1 \otimes \gamma_Q$ is a $\mathbb{Z}^n$-graded algebra isomorphism of $\mathcal{A}\otimes_k S$ onto $(\mathcal{A}\otimes_k S)_Q$. Consequently, to prove (26), it suffices to show that $1 \otimes \gamma_Q$ maps $\mathcal{B}$ onto $\mathcal{B}'$.

Now let $\bar{\Lambda} = \mathbb{Z}/(m_1) \oplus \ldots \oplus \mathbb{Z}/(m_n)$ and $\bar{\Lambda}' = \mathbb{Z}/(m_1') \oplus \ldots \oplus \mathbb{Z}/(m_n')$. We regard $\mathcal{A}$ as a $\bar{\Lambda}$-graded algebra with the grading determined by $\sigma_1, \dots, \sigma_n$ (see Definition 3.2.1). Let $\mathcal{A}'$ denote the algebra $\mathcal{A}$ with the $\bar{\Lambda}'$-grading determined by $\sigma_1', \dots, \sigma_n'$. Then $\mathcal{B} = \sum_{\boldsymbol{\ell}\in\mathbb{Z}^n} \mathcal{A}^{\bar{\boldsymbol{\ell}}} \otimes z^{\boldsymbol{\ell}}$ and so

$$(1 \otimes \gamma_Q)\mathcal{B} = \textstyle\sum_{\boldsymbol{\ell}\in\mathbb{Z}^n} \mathcal{A}^{\bar{\boldsymbol{\ell}}} \otimes z^{\boldsymbol{\ell} Q^t}.$$

On the other hand, $\mathcal{B}' = \sum_{\boldsymbol{\ell}\in\mathbb{Z}^n} \mathcal{A}'^{\bar{\boldsymbol{\ell}}} \otimes z^{\boldsymbol{\ell}}$, and so it suffices to show that

$$\mathcal{A}^{\bar{\boldsymbol{\ell}}} = \mathcal{A}'^{\overline{\boldsymbol{\ell} Q^t}} \tag{27}$$

for $\bar{\lambda} \in \bar{\Lambda}$. Since $\zeta_{ml}^m = \zeta_l$ for all $m, l \geq 1$, $\zeta(\frac{m}{n}) := \zeta_n^m$ defines a homomorphism $(\mathbb{Q}, +) \to k^\times$. Now

$$\mathcal{A}^{\bar{\boldsymbol{\ell}}} = \{x \in \mathcal{A} \mid gx = \chi_{\bar{\boldsymbol{\ell}}}(g)x \text{ for all } g \in G\},$$

where $\chi_{\bar{\boldsymbol{\ell}}}$ is the character on G with $\chi_{\bar{\boldsymbol{\ell}}}(\sigma_i) = \zeta_{m_i}^{l_i} = \zeta(\frac{l_i}{m_i})$. One checks that $\chi_{\bar{\boldsymbol{\ell}}}(\sigma^{\mathbf{k}}) = \zeta((\boldsymbol{\ell} D_{\mathbf{m}}^{-1}) \cdot \mathbf{k})$ where $\cdot$ is the usual dot product. Thus to prove (27) it is enough to show that

$$\chi_{\bar{\boldsymbol{\ell}}} = \chi'_{\overline{\boldsymbol{\ell} Q^t}}$$

where $\chi'_{\overline{\boldsymbol{\ell} Q^t}}(\sigma'^{\mathbf{k}}) = \zeta((\boldsymbol{\ell} Q^t D_{\mathbf{m}'}^{-1}) \cdot \mathbf{k})$. Since $\sigma'^{\mathbf{k}} = \sigma^{\mathbf{k}P^t}$ by (25), the result follows from

$$(\boldsymbol{\ell} D_{\mathbf{m}}^{-1}) \cdot (\mathbf{k} P^t) = (\boldsymbol{\ell} D_{\mathbf{m}}^{-1} P) \cdot \mathbf{k} = (\boldsymbol{\ell} Q^t D_{\mathbf{m}'}^{-1}) \cdot \mathbf{k}. \qquad \square$$

We next use the Correspondence Theorem to prove a second proposition about multiloop algebras.

Proposition 8.2.2.

(i) *Suppose that* $\mathbf{m} = (m_1, \dots, m_n)$ *is an* n*-tuple of positive integers,* $\mathcal{A}$ *is a central-simple (ungraded) algebra and* $\sigma_1, \dots, \sigma_n$ *are commuting algebra automorphisms of* $\mathcal{A}$ *such that* $\sigma^{m_i} = 1$ *for each* i*. Then* $M_{\mathbf{m}}(\mathcal{A}, \sigma_1, \dots, \sigma_n)$ *is a graded-central-simple* $\mathbb{Z}^n$*-graded algebra whose central grading group is given by*

$$\Gamma(M_{\mathbf{m}}(\mathcal{A}, \sigma_1, \dots, \sigma_n)) = m_1\mathbb{Z} \oplus \cdots \oplus m_n\mathbb{Z}. \tag{28}$$

(ii) *Suppose that* $\mathbf{m}$, $\mathcal{A}$ *and* $\sigma_1, \dots, \sigma_n$ *are as in (i) and* $\mathbf{m}'$, $\mathcal{A}'$ *and* $\sigma_1', \dots, \sigma_n'$ *are as in (i). Then* $M_{\mathbf{m}}(\mathcal{A}, \sigma_1, \dots, \sigma_n) \simeq_{\mathbb{Z}^n} M_{\mathbf{m}'}(\mathcal{A}', \sigma_1', \dots, \sigma_n')$ *if and only if* $\mathbf{m} = \mathbf{m}'$ *and there exists an algebra isomorphism* $\varphi : \mathcal{A} \to \mathcal{A}'$ *such that*

$$\varphi\sigma_j\varphi^{-1} = \sigma_j' \tag{29}$$

for $1 \leq j \leq n$.

Proof. (i) Let $\mathcal{B} = M_{\mathbf{m}}(\mathcal{A}, \sigma_1, \dots, \sigma_n)$ and $\Lambda = \mathbb{Z}^n$. Then, as we saw in Definition 3.2.1, we have $\mathcal{B} = L_\pi(\mathcal{A}, \Sigma)$, where $\bar{\Lambda} = \mathbb{Z}/(m_1) \oplus \cdots \oplus \mathbb{Z}/(m_n)$, $\pi : \Lambda \to \bar{\Lambda}$ is the natural map defined by (3), and Σ is the $\bar{\Lambda}$-grading on $\mathcal{A}$ determined by the automorphisms $\sigma_1, \dots, \sigma_n$. By part (i) of the Correspondence Theorem, $\mathcal{B}$ is graded-central-simple with central grading group $\ker(\pi)$. Since $\ker(\pi) = m_1\mathbb{Z} \oplus \cdots \oplus m_n\mathbb{Z}$, we have (28).

(ii): Let $\mathcal{B} = M_{\mathbf{m}}(\mathcal{A}, \sigma_1, \dots, \sigma_n)$, $\mathcal{B}' = M_{\mathbf{m}'}(\mathcal{A}', \sigma_1', \dots, \sigma_n')$ and $\Lambda = \mathbb{Z}^n$. If $\mathcal{B} \simeq_\Lambda \mathcal{B}'$, then $\Gamma(\mathcal{B}) = \Gamma(\mathcal{B}')$ and so we have $\mathbf{m} = \mathbf{m}'$ by (28). Consequently, for the rest of the proof of both directions in (ii), we can and do assume that $\mathbf{m} = \mathbf{m}'$.

Let $\bar{\Lambda}$ and π be as in the proof of (i), and let Σ (resp. Σ') be the $\bar{\Lambda}$-grading on $\mathcal{A}$ (resp $\mathcal{A}'$) determined by the automorphisms $\sigma_1, \dots, \sigma_n$ (resp. $\sigma_1', \dots, \sigma_n'$). Then $\mathcal{B} = L_\pi(\mathcal{A}, \Sigma)$ and $\mathcal{B}' = L_\pi(\mathcal{A}', \Sigma')$. Hence, by part (iii) of the Correspondence Theorem, it follows that $\mathcal{B} \simeq_\Lambda \mathcal{B}'$ if and only if $(\mathcal{A}, \Sigma) \sim_\pi (\mathcal{A}', \Sigma')$. Furthermore, since k is algebraically closed, we have by Remark 6.3.7(iii) that $(\mathcal{A}, \Sigma) \sim_\pi (\mathcal{A}', \Sigma')$ if and only if $(\mathcal{A}, \Sigma) \simeq_{\bar{\Lambda}} (\mathcal{A}', \Sigma')$. Consequently, it suffices to show that the $\bar{\Lambda}$-graded algebra isomorphisms from $(\mathcal{A}, \Sigma)$ to $(\mathcal{A}', \Sigma')$ are precisely the algebra isomorphisms $\varphi : \mathcal{A} \to \mathcal{A}'$ that satisfy (29). This is easily checked. □

8.3. **Multiloop realizations.**

Definition 8.3.1. Let $\mathfrak{B}(\Lambda, \mathrm{fi})$ be the class of Λ-graded algebras $\mathcal{B}$ such that $\mathcal{B}$ is graded-central-simple and $\Gamma(\mathcal{B})$ has finite index in Λ. Note that since Λ is free of finite rank (by assumption), it follows from Lemma 4.3.8 that any graded algebra $\mathcal{B}$ in $\mathfrak{B}(\Lambda, \mathrm{fi})$ has split centroid. Hence

$$\mathfrak{B}(\Lambda, \mathrm{fi}) = \cup_\Gamma \mathfrak{B}(\Lambda, \Gamma),$$

where the class union $\cup_\Gamma$ runs over all subgroups Γ of finite index in Λ.

Our next main result gives multiloop realizations and isomorphism conditions for all graded algebras in $\mathfrak{B}(\Lambda, \mathrm{fi})$. In this theorem we use the notion of *isograded-isomorphism* and the notation $\simeq_{\mathrm{ig}}$ described in Definition 2.1.3(ii).

Theorem 8.3.2 (Realization Theorem)**.** *Suppose that k is an algebraically closed field of characteristic 0.*

(i) *Suppose that $\mathcal{B}$ is a Λ-graded algebra, where Λ is a free abelian group of rank $n \geq 1$. Then $\mathcal{B} \in \mathfrak{B}(\Lambda, fi)$ if and only if $\mathcal{B}$ is isograded-isomorphic to $M_{\mathbf{m}}(\mathcal{A}, \sigma_1, \dots, \sigma_n)$ for some central-simple (ungraded) algebra $\mathcal{A}$, some n-tuple of positive integers $\mathbf{m} = (m_1, \dots, m_n)$ and some sequence $\sigma_1, \dots, \sigma_n$ of commuting finite order algebra automorphisms of $\mathcal{A}$ such that $\sigma^{m_i} = 1$ for all i.*

(ii) *Let $\Lambda = \mathbb{Z}^n$. Suppose $\mathcal{B} = M_{\mathbf{m}}(\mathcal{A}, \sigma_1, \dots, \sigma_n)$, where $\mathcal{A}$, $\mathbf{m}$, and $\sigma_1, \dots, \sigma_n$ are as in (i), and suppose $\mathcal{B}' = M_{\mathbf{m}'}(\mathcal{A}', \sigma_1', \dots, \sigma_n')$, where $\mathcal{A}'$, $\mathbf{m}'$, and $\sigma_1', \dots, \sigma_n'$ are as in (i). Then $\mathcal{B} \simeq_{ig} \mathcal{B}'$ if and only if there exists a matrix $P = (p_{ij}) \in \mathrm{GL}_n(\mathbb{Z})$ and an algebra isomorphism $\varphi : \mathcal{A} \to \mathcal{A}'$ such that P is $(\mathbf{m}', \mathbf{m})$-admissible and*

$$\sigma_j' = \varphi \left(\textstyle\prod_{i=1}^n \sigma_i^{p_{ij}} \right) \varphi^{-1} \tag{30}$$

for $1 \leq j \leq n$. Moreover, if $\langle \mathrm{supp}_{\mathbb{Z}^n}(\mathcal{B}) \rangle = \mathbb{Z}^n$ and $\langle \mathrm{supp}_{\mathbb{Z}^n}(\mathcal{B}') \rangle = \mathbb{Z}^n$, then $\mathcal{B} \simeq_{ig} \mathcal{B}'$ if and only if there exists a matrix $P = (p_{ij}) \in \mathrm{GL}_n(\mathbb{Z})$ and an algebra isomorphism $\varphi : \mathcal{A} \to \mathcal{A}'$ such that (30) holds for $1 \leq j \leq n$.

(iii) *Suppose* $\mathcal{B} \in \mathfrak{B}(\Lambda, fi)$*, where* Λ *is a free abelian group of rank* $n \geq 1$*, and suppose* $\mathcal{B} \simeq_{ig} M_{\mathbf{m}}(\mathcal{A}, \sigma_1, \ldots, \sigma_n)$*, where* $\mathcal{A}$*,* $\mathbf{m}$*, and* $\sigma_1, \ldots, \sigma_n$ *are as in (i). Then* $\mathcal{A}$ *is finite dimensional if and only if* $\mathcal{B}$ *is fgc.*

Proof. (i): The implication "$\Leftarrow$" follows from Proposition 8.2.2(i). To prove "$\Rightarrow$", let $\mathcal{B} \in \mathfrak{B}(\Lambda, \mathrm{fi})$. Then $\mathcal{B} \in \mathfrak{B}(\Lambda, \Gamma)$ for some subgroup Γ of Λ of finite index in Λ. Furthermore, by the fundamental theorem of finitely generated abelian groups, there exists a $\mathbb{Z}$-basis $\{ \lambda_1, \ldots, \lambda_n \}$ of Λ and an n-tuple $\mathbf{m} = (m_1, \ldots, m_n)$ of positive integers such that $m_i | m_{i+1}$ for $1 \leq i \leq n-1$ and $\Gamma = \langle m_1\lambda_1, \ldots, m_n\lambda_n \rangle$. We then identify $\Lambda = \mathbb{Z}^n$ is such a way that $\{ \lambda_1, \ldots, \lambda_n \}$ is the standard basis. (We can do this since we are working up to isomorphism of the grading groups.) Let $\bar{\Lambda} = \mathbb{Z}/(m_1) \oplus \cdots \oplus \mathbb{Z}/(m_n)$, and let $\pi : \Lambda \to \bar{\Lambda}$ be the natural map. Then π is an epimorphism with kernel Γ and so by Theorem 7.1.1(ii) there exists $(\mathcal{A}, \Sigma) \in \mathfrak{A}(\bar{\Lambda})$ such that $\mathcal{B} \simeq_\Lambda L_\pi(\mathcal{A}, \Sigma)$. (Here as in Definition 3.2.1, it is convenient to not abbreviate the graded algebra $(\mathcal{A}, \Sigma)$ as $\mathcal{A}$.) Now since $\Sigma = \{ \mathcal{A}^{\bar{\lambda}} \}_{\bar{\lambda} \in \bar{\Lambda}}$ is a $\bar{\Lambda}$-grading there exist unique algebra automorphisms $\sigma_1, \ldots, \sigma_n$ of $\mathcal{A}$ such that

$$\mathcal{A}^{(\bar{\ell}_1, \ldots, \bar{\ell}_n)} = \{ u \in \mathcal{A} \mid \sigma_j u = \zeta_{m_j}^{\ell_j} u \text{ for } 1 \leq j \leq n \}$$

for $(\ell_1, \ldots, \ell_n) \in \mathbb{Z}^n$. Then $\sigma_1, \ldots, \sigma_n$ is a sequence of commuting algebra automorphisms of $\mathcal{A}$ such that $\sigma_i^{m_i} = 1$ for all i. Furthermore, as we saw in Definition 3.2.1, we have $L_\pi(\mathcal{A}, \Sigma) = M_{\mathbf{m}}(\mathcal{A}, \sigma_1, \ldots, \sigma_n)$. Thus $\mathcal{B} \simeq_\Lambda M_{\mathbf{m}}(\mathcal{A}, \sigma_1, \ldots, \sigma_n)$.

(ii): By Remark 2.1.5, we have

$$\mathcal{B} \simeq_{\mathrm{ig}} \mathcal{B}' \iff \mathcal{B}' \simeq_{\mathbb{Z}^n} \mathcal{B}_\nu$$

for some $\nu \in \mathrm{Aut}(\mathbb{Z}^n)$.

Suppose for the moment that $\mathcal{B}' \simeq_{\mathbb{Z}^n} \mathcal{B}_\nu$, where $\nu \in \mathrm{Aut}(\mathbb{Z}^n)$. Then $\Gamma(\mathcal{B}') = \nu^{-1}(\Gamma(\mathcal{B}))$ (by Remark 4.2.5) and so $\nu(\Gamma(\mathcal{B}')) = \Gamma(\mathcal{B})$. So if we identify ν with a matrix $R \in \mathrm{GL}_n(\mathbb{Z})$ (as in (21)), we have $(\mathbb{Z}^n D_{\mathbf{m}'}) R^t = \mathbb{Z}^n D_{\mathbf{m}}$ by (28). Therefore R is $(\mathbf{m}', \mathbf{m})$-admissible by Lemma 8.1.2.

Consequently

$$\mathcal{B} \simeq_{\mathrm{ig}} \mathcal{B}' \iff \mathcal{B}' \simeq_{\mathbb{Z}^n} \mathcal{B}_R$$

for some $(\mathbf{m}', \mathbf{m})$-admissible matrix $R \in \mathrm{GL}_n(\mathbb{Z})$. The first statement in (ii) now follows from Proposition 8.2.1 and Proposition 8.2.2(ii).

To prove the second statement in (ii), suppose that $\langle \mathrm{supp}_{\mathbb{Z}^n}(\mathcal{B}) \rangle = \mathbb{Z}^n$ and $\langle \mathrm{supp}_{\mathbb{Z}^n}(\mathcal{B}') \rangle = \mathbb{Z}^n$. It suffices to show that if $\varphi : \mathcal{A} \to \mathcal{A}'$ is an algebra isomorphism and $P = (p_{ij}) \in \mathrm{GL}_n(\mathbb{Z})$ satisfies (30) for $1 \leq j \leq n$, then P is necessarily $(\mathbf{m}', \mathbf{m})$-admissible.

Now (30) implies that

$$\sigma'^{\boldsymbol{\ell}} = \varphi \sigma^{\boldsymbol{\ell} P^t} \varphi^{-1}$$

for $\boldsymbol{\ell} \in \mathbb{Z}^n$. Thus, there is an isomorphism of $\langle \sigma'_1, \ldots, \sigma'_n \rangle$ onto $\langle \sigma_1, \ldots, \sigma_n \rangle$ such that $\sigma'^{\boldsymbol{\ell}} \to \sigma^{\boldsymbol{\ell} P^t}$. Hence, by Lemma 3.2.4, there is an isomorphism from $\bar{\Lambda}'$ onto $\bar{\Lambda}$ such that $\bar{\boldsymbol{\ell}} \mapsto \bar{\boldsymbol{\ell}} P^t$. So, by Lemma 8.1.2, P is $(\mathbf{m}', \mathbf{m})$-admissible.

(iii): This follows from Theorem 7.1.1(iv). □

Definition 8.3.3. If $\mathcal{B}$ is a Λ-graded algebra, $\mathcal{A}$ is an algebra and $\mathbf{m}$ is an n-tuple of positive integers, we say that $\mathcal{B}$ has a *multiloop realization* based on $\mathcal{A}$ and relative to $\mathbf{m}$ if there exist commuting automorphisms $\sigma_1, \ldots, \sigma_n$ of $\mathcal{A}$ such that $\mathcal{B} \simeq_{\mathrm{ig}} M_{\mathbf{m}}(\mathcal{A}, \sigma_1, \ldots, \sigma_n)$.

Remark 8.3.4. Suppose that $\mathcal{B} \in \mathfrak{B}(\Lambda, \mathrm{fi})$. Then by the Realization Theorem (i), $\mathcal{B}$ has a multiloop algebra realization based on some central-simple algebra $\mathcal{A}$ and relative to some $\mathbf{m}$. We note now that more can be said about the choice of $\mathbf{m}$ and $\mathcal{A}$.

(i) By the proof of part (i) of the Theorem, we may choose $\mathbf{m}$ with the additional property that $m_i | m_{i+1}$ for $1 \le i \le n-1$. In that case, $\mathbf{m}$ is uniquely determined by $\mathcal{B}$. Indeed if $\mathcal{B}$ has a multiloop algebra realizations relative to both $\mathbf{m}$ and $\mathbf{m}'$, then by part (ii) of the Theorem there exists an $(\mathbf{m}', \mathbf{m})$-admissible matrix. But then by Lemma 8.1.2, $\mathbb{Z}/(m_1) \oplus \cdots \oplus \mathbb{Z}/(m_n) \simeq \mathbb{Z}/(m'_1) \oplus \cdots \oplus \mathbb{Z}/(m'_n)$. Hence if $m_i | m_{i+1}$ and $m'_i | m'_{i+1}$ for all i, we have $\mathbf{m} = \mathbf{m}'$ by the fundamental theorem for finitely generated abelian groups.

(ii) By part (ii) of the theorem, the ungraded algebra $\mathcal{A}$ is uniquely determined up to isomorphism by $\mathcal{B}$. Moreover, by the proof of part (i) of the theorem and by Theorem 7.1.1(ii), we may take $\mathcal{A}$ to be any central image of $\mathcal{B}$ (forgetting the grading on $\mathcal{A}$).

The Realization Theorem has the following corollary about multiloop realizations based on finite dimensional simple algebras.

Corollary 8.3.5. *Let $\mathcal{B}$ be a Λ-graded algebra. Then $\mathcal{B}$ has a multiloop realization based on a finite dimensional simple algebra if and only if $\mathcal{B}$ is graded-central-simple, $\mathcal{B}$ is fgc, and $\Gamma(\mathcal{B})$ has finite index in Λ. Moreover, if there exists a positive integer ℓ such that $\ell\Lambda \subseteq \mathrm{supp}_\Lambda(\mathcal{B})$, then $\mathcal{B}$ has a multiloop realization based on a finite dimensional simple algebra if and only if $\mathcal{B}$ is graded-central-simple and fgc.*

Proof. Since any finite dimensional simple algebra over the algebraically closed field k is central simple, the first statement follows from from parts (i) and (iii) of the Realization theorem. The second statement then follows from the implication "(a) $\Rightarrow$ (d)" in Proposition 4.4.5. □

Remark 8.3.6. In view of Proposition 4.4.5, the first statement in the preceding corollary can be stated alternatively as follows: $\mathcal{B}$ has a multiloop realization based on a finite dimensional simple algebra if and only if $\mathcal{B}$ is graded-central-simple, $\dim \mathcal{B}^\lambda < \infty$ for all $\lambda \in \Lambda$, and $\Gamma(\mathcal{B})$ has finite index in Λ.

9. Some classes of tori

The quantum torus discussed in Example 7.2.1 is an example of what is called an associative torus. (In fact this example explains the use of the term associative torus.) Furthermore there are nonassociative analogs of these algebras called alternative tori and Jordan tori, and there are Lie algebra analogs called Lie tori. When Λ is free of finite rank and $\mathrm{char}(k) = 0$, centreless Lie tori (Lie tori with trivial centre) play a basic role in the theory of extended affine Lie algebras because they appear as centreless cores of EALA's [26, 20]. Furthermore, associative, alternative and Jordan tori are also of great importance in this context because they arise as coordinate algebras of Lie tori of type A_ℓ (see [6], [7] and [25]). Thus an understanding of these classes of tori is very important in the theory of EALA's. In this section, we consider associative, alternative and Jordan tori, leaving Lie tori for a separate paper.

Initially, we suppose only that k is a field and Λ is an abelian group.

9.1. **Definitions.**

Definition 9.1.1. Suppose that $\mathcal{B}$ is a unital associative, alternative or Jordan Λ-graded algebra. (We assume that k has characteristic $\neq 2$ in the Jordan case.) Then $\mathcal{B}$ is said to be an *associative, alternative or Jordan Λ-torus* respectively if

(a) For each $\lambda \in \mathrm{supp}_\Lambda(\mathcal{B})$, $\mathcal{B}^\lambda$ is spanned by an invertible element of $\mathcal{B}$.
(b) $\langle \mathrm{supp}_\Lambda(\mathcal{B}) \rangle = \Lambda$.

(The interested reader can consult [27, §10.3 and §14.2] for the basic facts about invertibility in alternative and Jordan algebras.)

Proposition 9.1.2. *Suppose that $\mathcal{B}$ is an associative, alternative or Jordan Λ-torus. Then $\mathcal{B}$ is a graded-central-simple algebra and there exists a positive integer ℓ such that $\ell\Lambda \subseteq \mathrm{supp}_\Lambda(\mathcal{B})$. In fact, we may take $\ell = 1$ for associative and alternative tori, and $\ell = 2$ for Jordan tori.*

Proof. It is clear from the definition that $\mathcal{B}$ is graded-simple. Also $\dim_k(\mathcal{B}_\lambda) = 1$ for $\lambda \in \mathrm{supp}_\Lambda(\mathcal{B})$, and so by Lemma 4.3.4, $\mathcal{B}$ is graded-central-simple.

It remains to check the last statement in the proposition. Let $T = \mathrm{supp}_\lambda(\mathcal{B})$. If $\mathcal{B}$ is an associative or alternative torus, then T is a subgroup of Λ (since the product of two invertible elements is invertible) and so $T = \Lambda$. Suppose next that $\mathcal{B}$ is a Jordan torus. Then, $0 \in T$, $-T = T$ and $T + 2T \subseteq T$ (see [25, Lemma 3.5]). Since $\Lambda = \langle T \rangle$, it follows that $T + 2\Lambda \subseteq T$ and so $2\Lambda \subseteq T$ as desired. $\square$

9.2. **Multiloop realization of tori.** For the rest of the paper, we assume again that k is algebraically closed of characteristic 0 and Λ is free abelian of finite rank ≥ 1.

We have the following application of our results:

Theorem 9.2.1. *Suppose that k is an algebraically closed field of characteristic 0, and Λ is a free abelian group of finite rank ≥ 1. Suppose that $\mathcal{B}$ is an associative, alternative or Jordan Λ-torus. Then $\mathcal{B}$ has a multiloop realization based on a finite dimensional simple associative, alternative or Jordan algebra $\mathcal{A}$ if and only if $\mathcal{B}$ is fgc.*

Proof. This follows from Corollary 8.3.5 and Proposition 9.1.2. (See also Remark 3.1.2(iii).) $\square$

Remark 9.2.2. Let $\Lambda = \mathbb{Z}^n$ and $\mathcal{B} = M_{\mathbf{m}}(\mathcal{A}, \sigma_1, \ldots, \sigma_n)$, where $\mathbf{m} = (m_1, \ldots, m_n)$ is a sequence of positive integers, $\mathcal{A}$ is a finite dimensional simple associative, alternative or Jordan algebra, and $\sigma_1, \ldots, \sigma_n$ is a sequence of commuting automorphisms of $\mathcal{A}$ with $\sigma_i^{m_i} = 1$ for all i. Then, $\mathcal{B}$ is an associative, alternative or Jordan Λ-torus respectively if and only if the following conditions hold:

(a) The simultaneous eigenspaces in $\mathcal{A}$ for the automorphisms $\sigma_1, \ldots, \sigma_n$ are each spanned by an invertible element of $\mathcal{A}$.
(b) $|\langle \sigma_1, \ldots, \sigma_n \rangle| = m_1 \cdots m_n$.

We omit the verification of this observation which is straightforward using Lemma 3.2.4.

In view of Theorem 9.2.1 and Remark 9.2.2 (as well as the isomorphism condition in the last sentence of part (ii) of the Realization Theorem), the study of fgc associative, alternative or Jordan Λ-tori is equivalent to the study of sequences of commuting automorphisms of finite dimensional algebras satisfying the conditions (a) and (b) in Remark 9.2.2.

Remark 9.2.3. Proposition 9.1.2 is also true for centreless Lie tori (in fact one can again take $\ell = 2$). Hence, Theorem 9.2.1 is also true in that case. That is, a centreless Lie torus $\mathcal{L}$ over an algebraically closed field of characteristic 0 has a multiloop realization based on a finite dimensional simple algebra if and only if $\mathcal{L}$ is fgc. However, the analysis similar to Remark 9.2.2 is more subtle and requires a more detailed study of Lie tori. We therefore postpone further discussion of this topic to a sequel to this paper.

9.3. **A tensor product decomposition.** We now describe a tensor product decomposition for fgc associative tori. If m is a positive integer and e is an integer that is relatively prime to m, we use the notation $\mathcal{Q}(m,e)$ for the quantum torus determined by the 2×2-matrix $\left[\begin{smallmatrix} 1 & \zeta_m^e \\ \zeta_m^{-e} & 1 \end{smallmatrix}\right]$. $\mathcal{Q}(m,e)$ with its natural $\mathbb{Z}^2$-grading (see Example 7.2.1) is an associative $\mathbb{Z}^2$-torus. Also $k[z_1^{\pm 1},\dots,z_s^{\pm 1}]$ denotes the algebra of Laurent polynomials with its natural $\mathbb{Z}^s$-grading.

Proposition 9.3.1. *Suppose that k is algebraically closed of characteristic 0 and Λ is a free abelian group of finite rank $n \geq 1$. Let $\mathcal{B}$ be an fgc associative Λ-torus. Then*

$$\mathcal{B} \simeq_{ig} \mathcal{Q}(m_1,e_1)\otimes_k \dots \otimes_k \mathcal{Q}(m_r,e_r)\otimes_k k[z_1^{\pm 1},\dots,z_s^{\pm 1}], \tag{31}$$

where $r \geq 0$, $s \geq 0$, $n = 2r+s$, $m_1,\dots,m_r$ are integers ≥ 2 such that $m_1 \mid m_2 \mid \cdots \mid m_r$, and $e_1,\dots,e_r$ are integers so that $\gcd(e_i,m_i) = 1$ for all i. (On the right hand side of (31), the tensor product has the natural $\underbrace{\mathbb{Z}^2\oplus\cdots\oplus\mathbb{Z}^2}_{r}\oplus\mathbb{Z}^s$-grading determined by the gradings on the components.)

This proposition has been proved by K.-H. Neeb [19, Theorem III.3] using a normal form for skew-symmetric integral matrices and some additional arguments linking that normal form to the proposition (see Remark 9.3.2 below). In order to illustrate how our results can be used to deduce results about infinite dimensional graded algebras from results about finite dimensional graded algebras, we outline an alternate proof of the proposition using the Correspondence Theorem. (We use the Correspondence Theorem rather than the Realization Theorem, since the coordinate free point of view is more convenient here.)

Outline of a proof. Suppose that k, Λ and $\mathcal{B}$ are as in the proposition. Since $\operatorname{supp}(\mathcal{B}) = \Lambda$, it follows from Proposition 4.4.5 that $\Lambda/\Gamma(\mathcal{B})$ is finite. Let $\bar\Lambda = \Lambda/\Gamma(\mathcal{B})$ and let $\pi : \Lambda \to \bar\Lambda$ be the natural projection. Now by the Correspondence Theorem, $\mathcal{B} \simeq_\Lambda L_\pi(\mathcal{A})$ for some finite dimensional central-simple algebra $\mathcal{A}$ that is $\bar\Lambda$-graded. In fact $\mathcal{A}$ is an associative $\bar\Lambda$-torus. But, since k is algebraically closed, we can identify $\mathcal{A}$ with the algebra $M_\ell(k)$ of $\ell\times\ell$-matrices over k for some $\ell \geq 1$. Furthermore, gradings on $M_\ell(k)$ by abelian groups have been classified in [3]. Moreover, those for which $M_\ell(k)$ is a $\bar\Lambda$-torus have a particularly simple description [3, Theorem 5], which gives a tensor product decomposition for $\mathcal{A}$ analogous to (31) (without the divisibility condition $m_1 \mid m_2 \mid \cdots \mid m_r$). One can easily adjust this decomposition of $\mathcal{A}$ to get the divisibility condition. Then, in the final step of the proof one shows (arguing as in the fundamental theorem for finitely generated abelian groups) that the decomposition of $\mathcal{A}$ determines a decomposition of the loop algebra $L_\pi(\mathcal{A})$. □

Remark 9.3.2. In [19], Neeb proves more than is stated above. He shows that in (31) one can choose all $e_i = 1$ if $s > 0$ and one can choose $e_i = 1$ for $2 \leq i \leq r$ if

$s = 0$. This sharper result can be deduced from the one stated above by means of a change of variables (using the argument in the proof of [19, Theorem III.1]). We note also that Neeb's result in [19] is formulated in such a way that it holds over any base field k.

Remark 9.3.3. Alternative tori have been classified in general in [7]. Although we have not worked out the details, it should be straightforward to alternately apply our approach above to obtain a classification of all fgc alternative tori. This would use the description by A. Elduque of all gradings of the octonion algebra over k [11].

Jordan tori have been classified in general by Yoshii in [25]. Yoshii used deep results on prime Jordan algebras due to Zelmanov, and so a more elementary approach would also be interesting. Our approach to classifying fgc Jordan tori would require a description of the gradings on finite dimensional simple Jordan algebras that satisfy conditions (a) and (b) in Remark 9.2.2. Such a description does not seem yet to have been completed, although a lot is known about finite dimensional gradings (see for example [4] and the references therein).

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